Math 242 Homework Set #5 **Spring, 2004**

Assigned April 21. Due May 5.

The five problems have equal credit.

1. Let $\{a_n\}$ be a sequence of positive numbers such that $a_n \to 0$. Can we always partition the natural numbers into infinitely many disjoint, *infinite* subsets S_j ($j = 1, 2, 3, \ldots$) such that, for each j,

$$\sum_{n:n\in S_j} a_n < \infty?$$

Either prove that this can always be done, or exhibit a positive sequence $a_n \to 0$ for which it is impossible.

2. Let $g_n : [0,1] \mapsto \mathbf{R}$ be a sequence of non-negative continuous functions. Suppose that $g_n(x) \geq g_{n+1}(x)$ for all n and for all $x \in [0,1]$. Suppose furthermore that $g_n \to 0$ pointwise. Show that the infinite series

$$\sum_1^\infty (-1)^n g_n(x)$$

converges uniformly on $[0,1]$.

3. Let $f_n : [0,1] \mapsto \mathbf{R}$ be a sequence of continuous functions converging uniformly to some $f : [0,1] \mapsto \mathbf{R}$. Let $\{x_n\}$ be a sequence of points in $[0,1]$ such that $x_n \to$ some $\xi \in [0,1]$. Prove that $f_n(x_n) \to f(\xi)$.

4. Let $f_n : [0,1] \mapsto \mathbf{R}$ be a sequence of continuous functions converging merely *pointwise* to a function $f : [0,1] \mapsto \mathbf{R}$, and let $\{x_n\}$ and ξ be as in problem #3. Suppose in addition that the limit function f is continuous. Give an example to show that, even under these conditions, $f_n(x_n) \to f(\xi)$ still might not happen.

5. Let $\Omega \subset \mathbf{R}^d$ be open, and let f and g be differentiable functions mapping from Ω into $\mathbf{R}$. Define $h(x) = f(x)g(x)$. Show that h is also differentiable and that its gradient equals $f(x)\nabla g(x) + g(x)\nabla f(x)$.