A SEMI-DISCRETE LITTLEWOOD-PALEY INEQUALITY

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1. Introduction.

This paper treats the following rather general problem. Suppose that we have a family of real-valued functions $\{\phi_{(I)}\}_I$, defined on $\mathbf{R}^d$, indexed over the collection $\mathcal{D}$ of dyadic cubes $I \subset \mathbf{R}^d$. Each $\phi_{(I)}$ is smooth and satisfies

$$|\phi_{(I)}(x)| \leq |I|^{-1/2}(1 + |x - x_I|/\ell(I))^{-M} \tag{1.1}$$

$$|\nabla\phi_{(I)}(x)| \leq \ell(I)^{-1}|I|^{-1/2}(1 + |x - x_I|/\ell(I))^{-M-1} \tag{1.2}$$

for all $x \in \mathbf{R}^d$, where M is some *fixed* number strictly bigger than d; we are using $\ell(I)$ to denote I's sidelength and x_I to mean its center; as usual, we are using $|E|$ to denote the Lebesgue measure of a set E. Finally, each $\phi_{(I)}$ also has

$$\int \phi_{(I)} = 0. \tag{1.3}$$

It is well-known that the properties $(1.1) - (1.3)$ ensure the "almost-orthogonality" of the family $\{\phi_{(I)}\}_I$; i.e., the property that if

$$f(x) = \sum_I \lambda_I \phi_{(I)}(x) \tag{1.4}$$

($\{\lambda_I\}_I \subset \mathbf{R}$) is any finite sum, then

$$\int |f|^2\,dx \leq C(M, d) \sum_I |\lambda_I|^2. \tag{1.5}$$

In this paper we investigate the extent to which (1.5) remains true, or must be modified, when the Lebesgue measure on the left in (1.5) is replaced with a non-constant weight.

The most natural weighted generalization of (1.5) would be this: *For every non-negative* $v \in L^1_{loc}(\mathbf{R}^d)$, *there exist constants* $\nu(I, M, v)$ *such that, for every sum* (1.4),

$$\int |f|^2\,v\,dx \leq C(M, d) \sum_I |\lambda_I|^2 \nu(I, M, v). \tag{1.6}$$

In this paper we prove a version of (1.6) for arbitrary v (see Theorem 2.2 below). We obtain as a corollary that, if v belongs to the Muckenhoupt A_∞ class, then the constants $\nu(I, M, v)$ have an especially nice form: they can be taken to be

$$C(v, \epsilon)|I|^{-1} \int_{\mathbf{R}^d} \frac{v(x)}{(1 + |x - x_I|/\ell(I))^{2M-(d+\epsilon)}}\,dx \tag{1.7}$$

for any $\epsilon > 0$ (the constant $C(v, \epsilon)$ also depends on the A_∞ "parameters" of v). Recall that a weight v belongs to A_∞ if there exist positive constants a and b such that, for all cubes $I \subset \mathbf{R}^d$ and measurable subsets $E \subset I$,

$$\frac{\int_E v}{\int_I v} \leq a \left(\frac{|E|}{|I|}\right)^b.$$

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There are several equivalent characterizations of A_∞ (see [GR]). The one we will use in this paper is the following: There exist positive constants C and η such that, for all cubes I,

$$\int_I v(x)\,\log^\eta(e+v(x)/v_I))\,dx \le C\int_I v(x)\,dx,$$

where we are using v_I to denote v's average over I.

The reader should note that (1.7) is within shouting distance of best possible, since, if the sum in (1.6) had only one term, the exponent would be $2M$.

Our interest in inequalities like (1.6) is three-fold.

Singular integrals and wavelet expansions. Let $\{w_{(I)}\}_I$ be a family of smooth functions indexed on $\mathcal{D}$. We suppose that each $w_{(I)}$ is supported in τI, the τ-fold dilate of I (where $\tau \ge 1$ is fixed), and satisfies

$$\|w_{(I)}\|_\infty \le |I|^{-1/2}$$
$$\|\nabla w_{(I)}\|_\infty \le \ell(I)^{-1}|I|^{-1/2}$$
$$\int w_{(I)}\,dx = 0.$$

(Note that we do not require the $w_{(I)}$'s to be translates/dilates of each other.) Let $h = \sum_I \lambda_I w_{(I)}$ be a finite sum, and consider what happens when we convolve h with a Calderón-Zygmund kernel K. The product will not, in general, have the form $\sum_I \lambda_I w_{(I)}$, because the functions $K * w_{(I)}$ will not have compact support. However, it will be, up to a multiplicative constant, a sum like (1.4). Now, in many cases it is possible to write $K * h = \sum_I \tilde\lambda_I w_{(I)}$, where the $\tilde\lambda_I$'s depend in some messy (but linear) way on the λ_I's. We can then estimate the size of the $\tilde\lambda_I$'s, and use these estimates to bound $K * h$. These estimates, unfortunately, tend to be rather unsatisfactory (as Richard Wheeden and the author discovered in their work on Bergman spaces [WW1], where they used an approach like this). The reason is that the estimates one gets for the $\tilde\lambda_I$'s are of the form:

$$|\tilde\lambda_I| \le \sum_J c_{I,J}|\lambda_J|,$$

for certain positive constants $c_{I,J}$ (roughly speaking, $c_{I,J} \sim |\int w_{(I)}(K * w(J))|$; the relation is an equality when the $w_{(I)}$'s are an orthonormal basis). If we then wish to control $K*h$ in terms of the size of the original coefficients λ_I, we must resort to something like Schur's Lemma. This is extremely inefficient. The positivity of the $c_{I,J}$'s indicates that we are not making optimal use of the cancellation in the $K * w_{(I)}$'s. Indeed, so much gets thrown away that the necessary endpoint estimates for the Schur's Lemma interpolation (see (3.8)–(3.10) and the accompanying estimates from [WW1]) require an unacceptable amount of decay in the convolution kernel K. But this approach is unsatisfactory for another reason as well. In many applications, the original function h is *defined* by the coefficients λ_I; this was the situation in [WW1]. In such a case, understanding K's behavior really does mean getting a good bound on $K * h$ directly in terms of the λ_I's. The main theorems in section 2 provide such a bound.

Bergman-type inequalities. In [WW1], Richard Wheeden and the author studied the weighted-norm inequality

$$\left(\int_{\mathbf{R}^{d+1}_+} |\nabla u(x,y)|^q\,d\mu(x,y)\right)^{1/q} \le \left(\int_{\mathbf{R}^d} |f|^p\,\sigma\,dx\right)^{1/p} \tag{1.8}$$

where $u = P_y * f(x)$, the Poisson integral of f, $\sigma \in L^1_{loc}(\mathbf{R}^d)$ is a non-negative weight, μ is a Borel measure, and $1 < p \le q < \infty$. The authors looked for necessary and sufficient conditions on v and μ for the inequality (1.8) to hold for all f in a reasonable test class[1].

The authors approached (1.8) by first considering a dual form. Each derivative of u arises from convolution with a kernel $y^{-1}\psi_y$ that has smoothness, good (but not infinite!) decay, and cancellation. (Note: we

[1] E.g., in L^∞ and with compact support, or $\cup_{1\le p<\infty} L^p(\mathbf{R}^d)$.

are using $\psi_y(x) \equiv y^{-d}\psi(x/y)$ to denote the usual y-dilation of ψ.) For each such ψ, define the dual operator T_ψ,

$$T_\psi g(x) = \int_{\mathbf{R}_+^{d+1}} g(t,y)\, y^{-1}\psi_y(t-x)\, d\mu(t,y),$$

acting on bounded, compactly supported functions $g : \mathbf{R}_+^{d+1} \mapsto \mathbf{R}$. Let p' and q' be the dual exponents to p and q, and set $v = \sigma^{1-p'}$. Inequality (1.8) will hold if

$$\left(\int_{\mathbf{R}^d} |T_\psi g|^{p'}\, v\, dx\right)^{1/p'} \le C\left(\int_{\mathbf{R}_+^{d+1}} |g(t,y)|^{q'}\, d\mu(t,y)\right)^{1/q'} \tag{1.9}$$

holds for all g in our test class, and for each ψ.

For the moment, fix $p = q = 2$. We are about to make (1.9) look like (1.6). For each $I \in \mathcal{D}$, set $T(I) = I \times (\ell(I)/2, \ell(I)]$ (the usual "top half" of the Carleson box "over" I). Write

$$T_\psi g(x) = \sum_I \int_{T(I)} g(t,y)\, y^{-1}\psi_y(t-x)\, d\mu(t,y);$$

note that the sum is finite, because of g's compact support. We may now rewrite the sum in the form of (1.4),

$$T_\psi g(x) = \sum_I \lambda_I \phi_{(I)},$$

where each λ_I satisfies

$$|\lambda_I| \le C\left(\int_{T(I)} |g(t,y)|^2\, d\mu(t,y)\right)^{1/2} \mu(T(I))^{1/2}\ell(I)^{-1-d/2}.$$

Suppose we have (1.6). Then (1.9) will hold (for $p = q = 2$) if

$$\mu(T(I))^{1/2}\nu(I,M,v)^{1/2} \le c\ell(I)^{1+d/2} \tag{1.10}$$

for all dyadic cubes I. The reader should notice that, the better (i.e., larger) the exponent in (1.6), the stronger will be the corresponding sufficient condition (1.10). In [WW1], Richard Wheeden and the author attacked (1.9) via a Schur's Lemma argument, and the exponent they obtained, for general kernels ψ, was rather small: M, when $M \ge d+2$ and $p = q = 2$. Unfortunately, this did not quite cover the case of the Poisson integral (see below). Theorem 2.2 in the present paper gives an exponent of $2M - (d+\epsilon)$ when $p = q = 2$; in addition, it is valid for *any* $M > d$, and it *does* handle the Poisson integral. For the more general $L^p \mapsto L^q$ problem, the results of this paper yield an exponent of $p'M - (p'/2)(d+\epsilon)$, which is almost always an improvement over the general result (Theorem 3, page 936) from [WW1]. Why we don't always get an improvement remains a mystery.

The problem of limited decay. There are a number of theorems in analysis which state, "You can do so-and-so with this kernel, *if* it has 'enough' decay." One such result is the following, from [FKP] (Theorem 3.1):

Theorem FKP. *Suppose that w is a doubling weight on $\mathbf{R}^d$ with doubling constant ρ. There exists an $N_0 = N_0(\rho)$ such that $w \in A_\infty$ if and only if, for some constant C and some $\phi \in \mathcal{A}_{N_0}(\mathbf{R}^d)$ [defined below], with $\int \phi = 1$, and $\psi = \nabla\phi$, we have*

$$t^{-d}\int_0^t \int_{x:|x-x_0|<t} \frac{|\psi_s * w(x)|^2}{|\phi_s * w(x)|^2}\, dx\, \frac{ds}{s} \le C$$

for all $x_0 \in \mathbf{R}^d$ *and* $t > 0$.

The family $\mathcal{A}_{N_0}(\mathbf{R}^d)$ is a collection of "bump" functions, which Fefferman and Stein used to define their so-called "grand" maximal function in [FS]. To wit:

$$\mathcal{A}_{N_0}(\mathbf{R}^d) = \{\phi \in \mathcal{S}(\mathbf{R}^d) : \int_{\mathbf{R}^d} |D^\alpha \phi(x)|^2 (1+|x|)^{N_0}\, dx \le 1,\, |\alpha| \le N_0\}.$$

The number N_0 is liable to be pretty big. Indeed, the author at first considered using some variant of Theorem FKP in his attack on (1.6), but he gave that up when he saw no way to eliminate the requirement of excessive decay. Wheeden and the author encountered a like difficulty in their work in [WW1]. They had a general theorem which treated kernels that decayed at infinity to order $|x|^{-M}$, for $M \ge d+2$. Unfortunately, the convolution kernel that generates the y-derivative of $u = P_y * f$ only decays to order $|x|^{-d-1}$!. They had to concoct a special argument, using harmonicity and the semigroup property, to handle this term in ∇u. (Even with the argument, though, their result was not entirely satisfactory.)

The present paper attempts to fill this unmet (and, the author believes, only *apparently* arcane) need for good quadratic estimates on sums like (1.4), in which the functions $\phi_{(I)}$ decay at specific, limited rates.

The paper is laid out as follows. In section 2 we state and prove our main theorems, and we derive two corollaries for weighted Bergman-space inequalities of the type treated in [WW1]. In section 3 we state and prove a two-parameter analogue of the $L^2 \mapsto L^2$ case of our Theorem 2.2, and we give a corollary related to two-parameter Bergman-type inequalities.

2. The main theorems.

In this section, $\mathcal{D}$ is the family of dyadic cubes $I \subset \mathbf{R}^d$. The sidelength of any cube I is denoted by $\ell(I)$ and its center is x_I. If I is any cube and $\tau \ge 1$, then τI is the cube concentric with I and with sidelength $\tau\ell(I)$.

Our main theorem (Theorem 2.2) deals with families of smooth functions $\{\phi_{(I)}\}_I$, defined on $\mathbf{R}^d$, indexed over I, and satisfying (1.1) − (1.3), for some $M > d$ which only depdends on the family.

The statement of our main theorem requires one more technical definition.

Definition 2.1. *Let* $\eta > 0$. *If* $I \subset \mathbf{R}^d$ *is a cube and* $v \in L^1_{loc}(\mathbf{R}^d)$ *is non-negative, we set*

$$v(I,\eta) = \int_I v(x) \log^\eta(e + v(x)/v_I)\, dx,$$

where v_I *is* v*'s average over* I.

We shall follow the usual convention that, if v is a weight and $E \subset \mathbf{R}^d$ is measurable, then $v(E)$ denotes $\int_E v$; i.e., the 'v-measure' of E.

Theorem 2.2. *Let* $\{\phi_{(I)}\}_I$ *be a family satisfying* (1.1) − (1.3) *for some* $M > d$. *Let* $\eta > 1$ *and let* $2M - d > \epsilon > 0$. *There is a* $C = C(M, d, \eta, \epsilon)$ *such that for all* f *as in* (1.4) *and all non-negative* $v \in L^1_{loc}(\mathbf{R}^d)$,

$$\int_{\mathbf{R}^d} |f|^2\, v\, dx \le C \sum_I \frac{|\lambda_I|^2}{|I|} \sum_{j=0}^\infty 3^{-2Mj+(d+\epsilon)j} v(3^j I, \eta). \tag{2.1}$$

In particular, if $v \in A_\infty$,

$$\int_{\mathbf{R}^d} |f|^2\, v\, dx \le C'(M, d, \epsilon, v) \int_{\mathbf{R}^d} \left[\sum_I \frac{|\lambda_I|^2}{|I|}(1 + |x - x_I|/\ell(I))^{-(2M-(d+\epsilon))}\right] v\, dx. \tag{2.2}$$

The proof of Theorem 2.2 hinges on a decomposition due to Uchiyama (Lemma 3.5 from [U]) and two lemmas from [W1] and [W2]. Our Lemma 2.7 (see below) is essentially (2.8) from [W2]. The reader should notice that the only thing used in the proof of (2.8) in [W2] is the "goodness" (see the definition below) of the family of dyadic cubes.

Lemma 2.3. *Each $\phi_{(I)}$ can be decomposed*

$$\phi_{(I)} = C(M,d)\sum_{j=0}^{\infty} 3^{-Mj}\phi_{(I),j},$$

where each $\phi_{(I),j}$ is smooth and satisfies:

$$\begin{aligned} &\operatorname{supp}\phi_{(I),j} \subset 3^j I \\ &\|\phi_{(I),j}\|_\infty \leq |I|^{-1/2} \\ &\|\nabla\phi_{(I),j}\|_\infty \leq (3^j\ell(I))^{-1}|I|^{-1/2} \\ &\int \phi_{(I),j}\,dx = 0. \end{aligned}$$

Uchiyama proves his Lemma 3.5 for $M = d+1$, but it is easy to see that the proof goes through for any $M > d$. He also states it for 2^j's instead of 3^j's; what is important here is the lacunarity. We want 3^j's because of Lemma 2.5 below.

The reader should notice that our $\phi_{(I)}$'s differ from Uchiyama's by factors of $|I|^{1/2}$.

Definition 2.4. *A family of cubes $\mathcal{G}$ is said to be good if: a) for all Q and Q' in $\mathcal{G}$, either $Q \subset Q'$, $Q' \subset Q$, or $Q \cap Q' = \emptyset$; b) if Q and Q' belong to $\mathcal{G}$, $Q \subset Q'$, and $Q \neq Q'$, then $\ell(Q) \leq .5\ell(Q')$.*

Lemma 2.5. *Let m be an odd positive integer and let $\mathcal{F}$ be the family of all m-fold dilates of dyadic cubes. The family $\mathcal{F}$ can be decomposed*

$$\mathcal{F} = \cup_{i=1}^{m^d}\mathcal{G}_i,$$

where the $\mathcal{G}_i$'s are pairwise disjoint and each $\mathcal{G}_i$ is good.

Proof of Lemma 2.5: This is essentially Lemma 2.1 from [W1], which deals with $m = 3$ (see also [G], p. 416). The general case is similar.

It is enough to prove the lemma when $d = 1$. Note that, since m is odd, 2 is invertible in the ring $\mathbf{Z}/m\mathbf{Z}$. Let $[r]$ stand for the equivalence class of r modulo m. Abusing notation somewhat, we let $[2^{-k}]$ (for k positive) denote $[((m+1)/2)^k]$ (since $(m+1)/2$ is 2's multiplicative inverse). Let $\mathcal{F}_k$ denote those elements of $\mathcal{F}$ with sidelength $m2^{-k}$. Each $\mathcal{F}_k$ can be decomposed into m disjoint subfamilies $\tilde{\mathcal{G}}_k(s)$ $(0 \leq s < m)$, where each $I \in \tilde{\mathcal{G}}_k(s)$ has the form

$$I = [\frac{j}{2^k}, \frac{j+m}{2^k})$$

and $j \equiv s\ (m)$. A little computation shows that the right and left halves of such an I belong to $\tilde{\mathcal{G}}_{k+1}([2s])$, and that I is either the right of left half (depending on j's parity) of an interval in $\tilde{\mathcal{G}}_{k-1}([(m+1)s/2])$. Therefore, the desired families $\mathcal{G}_i$ $(0 \leq i < m)$ are

$$\mathcal{G}_i \equiv \cup_{k=-\infty}^{\infty}\tilde{\mathcal{G}}_k([[2^k]i]).$$

QED.

Definition 2.6. *Let $I \subset \mathbf{R}^d$ be a cube. A smooth function $a_{(I)}$ is said to be adapted to I if*

$$\begin{aligned} &\operatorname{supp} a_{(I)} \subset I \\ &\int a_{(I)} = 0 \\ &\|a_{(I)}\|_\infty \leq |I|^{-1/2} \\ &\|\nabla a_{(I)}\|_\infty \leq \ell(I)^{-1}|I|^{-1/2}. \end{aligned}$$

The lemma from [W2] is:

Lemma 2.7. *Let $\mathcal{G}$ be a good family of cubes in $\mathbf{R}^d$. Let $f = \sum_I \lambda_I a_{(I)}$ ($\lambda_I \in \mathbf{R}$) be a finite sum such that each $I \in \mathcal{G}$ and each $a_{(I)}$ is adapted to I. If $\eta > 1$ then*

$$\int |f|^2 \, v \, dx \le C(\eta, d) \sum_I \frac{|\lambda_I|^2}{|I|} v(I, \eta)$$

for all non-negative $v \in L^1_{loc}(\mathbf{R}^d)$.

We may now proceed to:

Proof of Theorem 2.2. For $j = 0, 1, 2, \ldots$, let $\mathcal{F}_j$ be the collection of all 3^j-fold dilates of dyadic cubes in $\mathbf{R}^d$. For each j, write $\mathcal{F}_j = \cup_{i=1}^{3^{jd}} \mathcal{G}(i,j)$, where the $\mathcal{G}(i,j)$'s are disjoint and good (as guaranteed by Lemma 2.5). Applying Lemma 2.3 to each $\phi_{(I)}$, we write:

$$\begin{aligned} f &= \sum_I \lambda_I \phi_{(I)} \\ &= C(M,d) \sum_I \lambda_I \sum_{j=0}^{\infty} 3^{-Mj} \phi_{(I),j} \\ &= C \sum_{j=0}^{\infty} \sum_{i=1}^{3^j d} \sum_{3^j I \in \mathcal{G}(i,j)} 3^{-Mj} \lambda_I \phi_{(I),j} \\ &= C \sum_{j=0}^{\infty} \sum_{i=1}^{3^j d} \sum_{3^j I \in \mathcal{G}(i,j)} 3^{-Mj} 3^{jd/2} \lambda_I \tilde{\phi}_{(I),j}, \end{aligned}$$

where $\tilde{\phi}_{(I),j} \equiv 3^{-jd/2} \phi_{(I),j}$. Notice that each $\tilde{\phi}_{(I),j}$ is adapted to $3^j I$. By Cauchy-Schwarz (twice),

$$\int_{\mathbf{R}^d} |f|^2 \, v \, dx \le C \sum_{j=0}^{\infty} 3^{j\epsilon + jd} \sum_{i=1}^{3^{jd}} \int_{\mathbf{R}^d} \Big| \sum_{3^j I \in \mathcal{G}(i,j)} 3^{-Mj} 3^{jd/2} \lambda_I \tilde{\phi}_{(I),j} \Big|^2 \, v \, dx.$$

However, by Lemma 2.7,

$$\begin{aligned} \int \Big| \sum_{3^j I \in \mathcal{G}(i,j)} 3^{-Mj} 3^{jd/2} \lambda_I \tilde{\phi}_{(I),j} \Big|^2 \, v \, dx &\le C(\eta, d) \sum_{3^j I \in \mathcal{G}(i,j)} 3^{-2Mj} 3^{jd} \frac{|\lambda_I|^2}{|3^j I|} v(3^j I, \eta) \\ &= C(\eta, d) \sum_{3^j I \in \mathcal{G}(i,j)} 3^{-2Mj} \frac{|\lambda_I|^2}{|I|} v(3^j I, \eta). \end{aligned}$$

Plugging this back into the preceding inequality finishes the proof of (2.1).

To prove (2.2), we note that, when $v \in A_\infty$, $v(I, \eta) \le C(\eta, v) \int_I v$ for all cubes I. Thus, by (2.1),

$$\begin{aligned} \int_{\mathbf{R}^d} |f|^2 \, v \, dx &\le C \sum_I \frac{|\lambda_I|^2}{|I|} \sum_{j=0}^{\infty} 3^{-2Mj + (d+\epsilon)j} v(3^j I, \eta) \\ &\le C \sum_I \frac{|\lambda_I|^2}{|I|} \sum_{j=0}^{\infty} 3^{-2Mj + (d+\epsilon)j} \int_{3^j I} v \, dx \\ &\le C \int_{\mathbf{R}^d} \left[\sum_I \frac{|\lambda_I|^2}{|I|} (1 + |x - x_I|/\ell(I))^{-(2M - (d+\epsilon))} \right] v \, dx. \end{aligned}$$

QED.

Corollary 2.8. *Let M, d, and ϵ be as in Theorem 2.2 Let ψ be a smooth function satisfying*

$$|\psi(x)| \le (1+|x|)^{-M}$$
$$|\nabla\psi(x)| \le (1+|x|)^{-M-1}$$
$$\int_{\mathbf{R}^d} \psi = 0.$$

Consider the Bergman-type inequality

$$\left(\int_{\mathbf{R}^{d+1}_+} |y^{-1}\psi_y * f(x)|^q \, d\mu(x,y)\right)^{1/q} \le \left(\int_{\mathbf{R}^d} |f|^p \, v \, dx\right)^{1/p}, \tag{2.3}$$

where $v \in L^1_{loc}(\mathbf{R}^d)$ is non-negative, μ is a Borel measure, and f belongs to a reasonable test class. Let $1 < p \le 2 \le q < \infty$, and set $\sigma = v^{1-p'}$, where p' is p's dual exponent. Let $\tau > p'/2$. There is a constant $c = c(M, d, \epsilon, p, q, \tau)$ such that (2.3) will hold for all f if there is a weight w such that

$$\sigma(I,\tau) \le \int_I w$$

and

$$\mu(T(I))^{1/q}\left(\int_{\mathbf{R}^d} \frac{w(x)}{(1+|x-x_I|/\ell(I))^{p'M-(p'/2)(d+\epsilon)}}\, dx\right)^{1/p'} \le c\ell(I)^{d+1}$$

for all cubes I.

Proof of Corollary 2.8. Following the pattern of [WW1], we consider the dual form of (2.3):

$$\left(\int_{\mathbf{R}^d} |Tg|^{p'} \, \sigma \, dx\right)^{1/p'} \le \left(\int_{\mathbf{R}^{d+1}_+} |g(t,y)|^{q'} \, d\mu(t,y)\right)^{1/q'},$$

where g is bounded and has compact support in $\mathbf{R}^{d+1}_+$; and

$$Tg(x) = \int_{\mathbf{R}^{d+1}_+} g(t,y)\, y^{-1}\psi_y(t-x)\, d\mu(t,y).$$

Write

$$\begin{aligned} Tg(x) &= \sum_I \int_{T(I)} g(t,y)\, y^{-1}\psi_y(t-x)\, d\mu(t,y) \\ &= \sum_I \lambda_I \phi_{(I)}(x), \end{aligned} \tag{2.4}$$

where $T(I)$ is the usual top half of a Carleson box, each $\phi_{(I)}$ is as in the (2.1), and the λ_I's satisfy

$$|\lambda_I| \le c\left(\int_{T(I)} |g|^{q'} \, d\mu\right)^{1/q'} \mu(T(I))^{1/q}\ell(I)^{-(1+d/2)}.$$

Note that, because of g's compact support, the sum (2.4) is finite.

Let ρ be the dual exponent to $p'/2$, and let h be non-negative and satisfy $\int |h|^\rho \, \sigma = 1$.

We wish to estimate $\int_{\mathbf{R}^d} |Tg|^2 \, h \, \sigma \, dx$.

Let η be a number bigger than 1, to be chosen presently. By (2.1),

$$\int_{\mathbf{R}^d} |Tg|^2 \, h \, \sigma \, dx \le C \sum_I \frac{|\lambda_I|^2}{|I|} \sum_{j=0}^{\infty} 3^{-2Mj+(d+\epsilon/2)j}(h\sigma)(3^j I, \eta). \tag{2.5}$$

(The '$\epsilon/2$' is not a typo: see below.) By virtue of a trick from [W3] (see inequality (13) there),

$$(h\sigma)(3^jI,\eta) \le C\sigma(3^jI,\eta p'/2)^{2/p'}.$$

Choose $\eta = \tau/(p'/2) > 1$. Then

$$\sigma(3^jI,\tau) \le \int_{3^jI} w \equiv w(3^jI).$$

Therefore

$$\int_{\mathbf{R}^d} |Tg|^2\, h\,\sigma\, dx \le C\sum_I \frac{|\lambda_I|^2}{|I|} \sum_{j=0}^{\infty} 3^{-2Mj+(d+\epsilon/2)j} w(3^jI)^{2/p'}.$$

Now,

$$\begin{aligned}\sum_{j=0}^{\infty} 3^{-2Mj+(d+\epsilon/2)j} w(3^jI)^{2/p'} &\le C_{\epsilon,p'}\left(\sum_{j=0}^{\infty} 3^{(-2Mj+(d+\epsilon)j)p'/2} w(3^jI)\right)^{2/p'}\\ &\le C\left(\int_{\mathbf{R}^d} \frac{w(x)}{(1+|x-x_I|/\ell(I))^{p'M-(p'/2)(d+\epsilon)}}\, dx\right)^{2/p'}.\end{aligned}$$

Thus, the right-hand side of (2.5) is less than or equal to

$$\begin{aligned}&C\sum_I \frac{|\lambda_I|^2}{|I|}\left(\int_{\mathbf{R}^d} \frac{w(x)}{(1+|x-x_I|/\ell(I))^{p'M-(p'/2)(d+\epsilon)}}\, dx\right)^{2/p'}\\ \le &C\sum_I\left(\int_{T(I)} |g|^{q'}\, d\mu\right)^{2/q'} \mu(T(I))^{2/q}\ell(I)^{-(2d+2)}\left(\int_{\mathbf{R}^d} \frac{w(x)}{(1+|x-x_I|/\ell(I))^{p'M-(p'/2)(d+\epsilon)}}\, dx\right)^{2/p'}. \quad (2.6)\end{aligned}$$

Our hypothesis on μ and w is that

$$\mu(T(I))^{1/q}\left(\int_{\mathbf{R}^d} \frac{w(x)}{(1+|x-x_I|/\ell(I))^{p'M-(p'/2)(d+\epsilon)}}\, dx\right)^{1/p'} \le c\ell(I)^{d+1}.$$

Therefore, the right-hand side of (2.6) is less than or equal to

$$C\sum_I\left(\int_{T(I)} |g|^{q'}\, d\mu\right)^{2/q'} \le C\left(\int_{\mathbf{R}^{d+1}_+} |g(t,y|^{q'}\, d\mu(t,y)\right)^{2/q'},$$

where the inequality follows because $2/q' \ge 1$. This proves the corollary. QED.

Theorem 2.2 is an $L^2 \mapsto L^2$ result. By carefully examining certain of the proofs from [W1], we can see how the (essential) conclusion of Theorem 2.2 may be reshaped into an $L^p \mapsto L^p$ result.

Definition 2.9. *For $0 < p < \infty$, define*

$$\tilde{c}(p) = \begin{cases} 2/p - 1 & \text{if } 0 < p \le 1;\\ 1 & \text{if } 1 < p \le 2;\\ 2/p' & \text{if } p > 2.\end{cases}$$

Remark: Notice that $\tilde{c}(p)$ is a continuous function of p and that it is always ≥ 1.

Theorem 2.10. *Let $0 < p < \infty$ and let $\eta > p/2$. Suppose that $M > d$ and $2M - d\tilde{c}(p) > \epsilon > 0$. There is a constant $C = C(M, \epsilon, \eta, p, d)$ such if v and w are non-negative functions in $L^1_{loc}(\mathbf{R}^d)$ satisfying*

$$v(I, \eta) \leq w(I)$$

for all cubes I, then

$$\int_{\mathbf{R}^d} |f|^p \, v \, dx \leq C \int_{\mathbf{R}^d} \left[\sum_I \frac{|\lambda_I|^2}{|I|} \left(1 + |x - x_I|/\ell(I)\right)^{-2M + (d\tilde{c}(p) + \epsilon)} \right]^{p/2} w \, dx$$

for all finite sums (1.4).

Proof of Theorem 2.10. The proof closely follows the lines of the proof of Theorem 2.2, the chief difference being that here we use a somewhat generalized form of Lemma 2.7; to wit:

Lemma 2.11. *Let $\mathcal{G}$ be a good family of cubes in $\mathbf{R}^d$. Let $f = \sum_I \lambda_I a_{(I)}$ ($\lambda_I \in \mathbf{R}$) be a finite sum such that each $I \in \mathcal{G}$ and each $a_{(I)}$ is adapted to I. Suppose that $0 < p < \infty$, $\eta > p/2$, and v and w are weights in $L^1_{loc}(\mathbf{R}^d)$ satisfying*

$$v(I, \eta) \leq w(I) \tag{2.7}$$

for all cubes I. There is a constant $C = C(\eta, p, d)$ such that

$$\int_{\mathbf{R}^d} |f|^p \, v \, dx \leq C \int_{\mathbf{R}^d} \left(\sum \frac{|\lambda_I|^2}{|I|} \chi_I(x) \right)^{p/2} w \, dx.$$

Proof of Lemma 2.11. For $0 < p \leq 2$, this is done explicitly in [W1] (Theorem 2.5). For $p \geq 2$, it is done implicitly in the proof of Theorem 2.7 there. Using the notation from that paper, set $\aleph(j) = 2^{\tau j}$, where $\tau > 0$ is small and will be chosen presently. Then our desired norm inequality will hold if

$$\left(\frac{\int_I v(x) \log(e + v(x)/v_I) \, dx}{v(I)} \right)^{\tau + p/2} v(I) \leq w(I)$$

for all cubes I. By Hölder's inequality, this will happen if (2.7) holds and $\tau + p/2 \leq \eta$. This proves the lemma.

To continue with proof of the theorem, we distinguish three cases: *a)* $0 < p \leq 1$; *b)* $1 < p \leq 2$; *c)* $p > 2$.
Case a): $0 < p \leq 1$.

Proceeding as in the proof of Theorem 2.2, we write:

$$\begin{aligned} f &= \sum_I \lambda_I \phi_{(I)} \\ &= C(M, d) \sum_I \lambda_I \sum_{j=0}^{\infty} 3^{-Mj} \phi_{(I),j} \\ &= C \sum_{j=0}^{\infty} \sum_{i=1}^{3^j d} \sum_{3^j I \in \mathcal{G}(i,j)} 3^{-Mj} \lambda_I \phi_{(I),j} \\ &= C \sum_{j=0}^{\infty} \sum_{i=1}^{3^j d} \sum_{3^j I \in \mathcal{G}(i,j)} 3^{-Mj} 3^{jd/2} \lambda_I \tilde{\phi}_{(I),j}, \end{aligned}$$

where everything has the same meaning here that it had in the earlier proof. Thus:

$$\begin{aligned} |f|^p &\leq C_\epsilon \sum_{j=0}^{\infty} 3^{j\epsilon} \Big| \sum_{i=1}^{3^j d} \sum_{3^j I \in \mathcal{G}(i,j)} 3^{-Mj} 3^{jd/2} \lambda_I \tilde{\phi}_{(I),j} \Big|^p \\ &\leq C_\epsilon \sum_{j=0}^{\infty} 3^{j\epsilon} \sum_{i=1}^{3^j d} \Big| \sum_{3^j I \in \mathcal{G}(i,j)} 3^{-Mj} 3^{jd/2} \lambda_I \tilde{\phi}_{(I),j} \Big|^p, \end{aligned}$$

where the second inequality follows because $p \leq 1$.

Temporarily fix i and j. Because of Lemma 2.11 and our hypotheses on v and w,

$$\int_{\mathbf{R}^d} |\sum_{I:3^j I\in\mathcal{G}(i,j)} 3^{-Mj}3^{jd/2}\lambda_I\tilde{\phi}_{(I),j}|^p\, v\, dx \leq C(p,M,d,\eta)\int_{\mathbf{R}^d}\left(\sum_{3^j I\in\mathcal{G}(i,j)}\frac{|\lambda_I|^2}{|I|}3^{-2Mj}\chi_{3^j I}\right)^{p/2} w\, dx.$$

Now, with j still fixed, we sum on i from 1 to 3^{jd}. Recall that, if $0 < r \leq 1$ and $a_k \geq 0$, then

$$\sum_1^N a_k^r \leq \left(\sum_1^N a_k\right)^r N^{1-r}.$$

In our case, $r = p/2$ and $N = 3^{jd}$. Thus:

$$\begin{aligned}\int_{\mathbf{R}^d}\sum_{i=1}^{3^{jd}} |\sum_{I:3^j I\in\mathcal{G}(i,j)} 3^{-Mj}3^{jd/2}\lambda_I\tilde{\phi}_{(I),j}|^p\, v\, dx &\leq C(p,M,d,\eta)\int_{\mathbf{R}^d} 3^{jd(1-p/2)}\left(\sum_I\frac{|\lambda_I|^2}{|I|}3^{-2Mj}\chi_{3^j I}\right)^{p/2} w\, dx\\ &= C(p,M,d,\eta)\int_{\mathbf{R}^d}\left(\sum_I\frac{|\lambda_I|^2}{|I|}3^{-2Mj+(2/p-1)jd}\chi_{3^j I}\right)^{p/2} w\, dx\\ &= C(p,M,d,\eta)\int_{\mathbf{R}^d}\left(\sum_I\frac{|\lambda_I|^2}{|I|}3^{-2Mj+\tilde{c}(p)jd}\chi_{3^j I}\right)^{p/2} w\, dx.\end{aligned}$$

Now sum on j for $j = 0$ to ∞. At the expense of having a slightly larger ϵ, we end up with:

$$\int_{\mathbf{R}^d}|f|^p\, v\, dx \leq C(\epsilon,p,M,d,\eta)\int_{\mathbf{R}^d}\left(\sum_I\frac{|\lambda_I|^2}{|I|}\sum_{j=0}^{\infty}3^{j(-2M+\tilde{c}(p)d+\epsilon)}\chi_{3^j I}\right)^{p/2} w\, dx.$$

It is easy to see that

$$\sum_{j=0}^{\infty}3^{j(-2M+\tilde{c}(p)d+\epsilon)}\chi_{3^j I}(x) \leq C(1+|x-x_I|/\ell(I))^{-2M+\tilde{c}(p)d+\epsilon}.$$

This finishes the proof in case a).

Case b): $1 < p \leq 2$.

We begin somewhat as we did with case a), writing:

$$\begin{aligned}|f|^p &\leq C_\epsilon\sum_{j=0}^{\infty}3^{j\epsilon}|\sum_{i=1}^{3^j d}\sum_{3^j I\in\mathcal{G}(i,j)} 3^{-Mj}3^{jd/2}\lambda_I\tilde{\phi}_{(I),j}|^p\\ &\leq C_\epsilon\sum_{j=0}^{\infty}3^{j\epsilon}3^{jdp/p'}\sum_{i=1}^{3^j d}|\sum_{3^j I\in\mathcal{G}(i,j)} 3^{-Mj}3^{jd/2}\lambda_I\tilde{\phi}_{(I),j}|^p,\end{aligned}$$

where the second inequality now follows from Hölder's inequality. We rewrite the right-hand side of this inequality as:

$$C_\epsilon\sum_{j=0}^{\infty}3^{j\epsilon}\sum_{i=1}^{3^j d}|\sum_{3^j I\in\mathcal{G}(i,j)} 3^{-Mj+jd/p'}3^{jd/2}\lambda_I\tilde{\phi}_{(I),j}|^p.$$

Once again, we fix i and j, and we apply Lemma 2.11. We obtain:

$$\int_{\mathbf{R}^d} |\sum_{3^j I\in\mathcal{G}(i,j)} 3^{-Mj+jd/p'}3^{jd/2}\lambda_I\tilde{\phi}_{(I),j}|^p\, v\,dx \le C(M,d,p,\eta)\int_{\mathbf{R}^d}\left(\sum_{3^j I\in\mathcal{G}(i,j)}\frac{|\lambda_I|^2}{|I|}3^{-2Mj+2jd/p'}\chi_{3^jI}\right)^{p/2} w\,dx.$$

Now, still keeping j fixed, we sum on i from 1 to 3^jd. Since $p/2\le 1$, we get:

$$\int_{\mathbf{R}^d}\sum_{i=1}^{3^{jd}}|\sum_{I:3^j I\in\mathcal{G}(i,j)} 3^{-Mj}3^{jd/2}\lambda_I\tilde{\phi}_{(I),j}|^p\, v\,dx \le C(p,M,d,\eta)\int_{\mathbf{R}^d}3^{jd(1-p/2)}\left(\sum_I\frac{|\lambda_I|^2}{|I|}3^{-2Mj+2jd/p'}\chi_{3^jI}\right)^{p/2} w\,dx$$

$$= C(p,M,d,\eta)\int_{\mathbf{R}^d}\left(\sum_I\frac{|\lambda_I|^2}{|I|}3^{-2Mj+(2/p-1)jd+2jd/p'}\chi_{3^jI}\right)^{p/2} w\,dx.$$

We note that, for $1<p\le 2$, $2/p-1+2/p'=1=\tilde{c}(p)$ (!). Therefore, we may rewrite the last quantity as:

$$C(p,M,d,\eta)\int_{\mathbf{R}^d}\left(\sum_I\frac{|\lambda_I|^2}{|I|}3^{-2Mj+\tilde{c}(p)jd}\chi_{3^jI}\right)^{p/2} w\,dx.$$

The rest of the proof proceeds exactlly as in case a).

Case c): $p>2$.

We begin as with case b):

$$|f|^p \le C_\epsilon\sum_{j=0}^{\infty}3^{j\epsilon}|\sum_{i=1}^{3^jd}\sum_{3^jI\in\mathcal{G}(i,j)}3^{-Mj}3^{jd/2}\lambda_I\tilde{\phi}_{(I),j}|^p$$

$$\le C_\epsilon\sum_{j=0}^{\infty}3^{j\epsilon}3^{jdp/p'}\sum_{i=1}^{3^jd}|\sum_{3^jI\in\mathcal{G}(i,j)}3^{-Mj}3^{jd/2}\lambda_I\tilde{\phi}_{(I),j}|^p$$

$$= C_\epsilon\sum_{j=0}^{\infty}3^{j\epsilon}\sum_{i=1}^{3^jd}|\sum_{3^jI\in\mathcal{G}(i,j)}3^{-Mj+jd/p'}3^{jd/2}\lambda_I\tilde{\phi}_{(I),j}|^p.$$

Fixing i and j, and applying Lemma 2.11, we get:

$$\int_{\mathbf{R}^d} |\sum_{3^j I\in\mathcal{G}(i,j)} 3^{-Mj+jd/p'}3^{jd/2}\lambda_I\tilde{\phi}_{(I),j}|^p\, v\,dx \le C(M,d,p,\eta)\int_{\mathbf{R}^d}\left(\sum_{3^j I\in\mathcal{G}(i,j)}\frac{|\lambda_I|^2}{|I|}3^{-2Mj+2jd/p'}\chi_{3^jI}\right)^{p/2} w\,dx.$$

We now sum on i from 1 to 3^{jd}. We use the fact that, if $r>1$ and $a_k\ge 0$, then

$$\sum a_k^r \le \left(\sum a_k\right)^r.$$

For us, $r=p/2>1$. Thus, with j still fixed:

$$\int_{\mathbf{R}^d}\sum_{i=1}^{3^{jd}}|\sum_{I:3^j I\in\mathcal{G}(i,j)} 3^{-Mj}3^{jd/2}\lambda_I\tilde{\phi}_{(I),j}|^p\, v\,dx \le C(p,M,d,\eta)\int_{\mathbf{R}^d}\left(\sum_I\frac{|\lambda_I|^2}{|I|}3^{-2Mj+2jd/p'}\chi_{3^jI}\right)^{p/2} w\,dx$$

$$= C(p,M,d,\eta)\int_{\mathbf{R}^d}\left(\sum_I\frac{|\lambda_I|^2}{|I|}3^{-2Mj+2jd/p'}\chi_{3^jI}\right)^{p/2} w\,dx$$

$$= C(p,M,d,\eta)\int_{\mathbf{R}^d}\left(\sum_I\frac{|\lambda_I|^2}{|I|}3^{-2Mj+\tilde{c}(p)jd}\chi_{3^jI}\right)^{p/2} w\,dx$$

.

The proof now concludes as with the two preceding cases. Theorem 2.10 is proved. QED.

Remarks. The conclusion of Theorem 2.10 should be distinguished from the chief result from [W4]. That paper treats linear sums

$$f = \sum_I \lambda_I \phi_{(I)}$$

in which the functions $\phi_{(I)}$, indexed over the family $\mathcal{D}$, are assumed to satisfy

$$|\phi_{(I)}(x)| + \ell(I)|\nabla\phi_{(I)}(x)| \leq |I|^{-1/2}(1 + |x - x_I|/\ell(I))^{-M} \tag{2.8a}$$

for some $M > d/2$ (although, in practice, one usually requires $M > d$), along with an *a priori* almost-orthogonality condition; namely, that for all finite linear sums $\sum_I \gamma_I \phi_{(I)}$, one has

$$\int |\sum_I \gamma_I \phi_{(I)}|^2 \, dx \leq \sum_I |\gamma_I|^2. \tag{2.8b}$$

The result obtained in that paper is:

Theorem 2.12. *Let $\tilde{\mathcal{F}} = \{\phi_{(I)}\}_I$ be a family of functions satisfying* (2.8*a*) *and* (2.8*b*). *Let $\sigma \in L^1_{loc}(\mathbf{R}^d)$ belong to the Muckenhoupt A_∞ class. Suppose that $2M - d > \epsilon > 0$, and $0 < p < \infty$. There is a constant $C = C(\sigma, \epsilon, p)$ such that, for all finite sums $f = \sum_I \lambda_I \phi_{(I)}$,*

$$\int_{\mathbf{R}^d} |f|^p \, \sigma \, dx \leq C \int_{\mathbf{R}^d} (g^*(x))^p \, \sigma \, dx,$$

where

$$g^*(x) = \left(\sum_I \frac{|\lambda_I|^2}{|I|} (1 + |x - x_I|/\ell(I))^{-2M+(d+\epsilon)} \right)^{1/2}.$$

Note that the "decay exponent" in the definition of g^* is $-2M + d + \epsilon$, valid for all p; whereas the corresponding quantity in Theorem 2.10 is $-2M + \tilde{c}(p)d + \epsilon$. We observed earlier that $\tilde{c}(p) \geq 1$ always, with equality only when $1 \leq p \leq 2$. Theorem 2.12 also requires weaker hypotheses on the family of $\phi_{(I)}$'s than does Theorem 2.10. Thus, for A_∞ weights, Theorem 2.12 gives a stronger result. The author finds it surprising that Theorem 2.10, in which the analysis is relatively soft, gives as good an exponent as Theorem 2.12 does even for the range $1 \leq p \leq 2$.

With the help of Theorem 2.10, we can now prove sufficient conditions for Bergman-space inequalities in the full range of p's and q's treated in [WW1].

Corollary 2.13. *Let M, d and ϵ be as in Theorem 2.2. Let ψ be a smooth function satisfying*

$$|\psi(x)| \leq (1 + |x|)^{-M}$$
$$|\nabla\psi(x)| \leq (1 + |x|)^{-M-1}$$
$$\int_{\mathbf{R}^d} \psi = 0.$$

Consider the Bergman-type inequality

$$\left(\int_{\mathbf{R}^{d+1}_+} |y^{-1}\psi_y * f(x)|^q \, d\mu(x, y) \right)^{1/q} \leq \left(\int_{\mathbf{R}^d} |f|^p \, v \, dx \right)^{1/p}, \tag{2.9}$$

where $v \in L^1_{loc}(\mathbf{R}^d)$ is non-negative, μ is a Borel measure, and f belongs to a reasonable test class. Let $2 < p \leq q < \infty$, and set $\sigma = v^{1-p'}$, where p' is p's dual exponent. In order that (2.9) *should hold for all f, it is sufficient that there exist a number $\tau > p'/2$ and a weight w such that*

$$\sigma(I, \tau) \leq \int_I w$$

and

$$\mu(T(I))^{1/q}\left(\int_{\mathbf{R}^d}\frac{w(x)}{(1+|x-x_I|/\ell(I))^{p'M-(p'/2)(d+\epsilon)}}\,dx\right)^{1/p'}\le c\ell(I)^{d+1}$$

for all cubes I.

Proof of Corollary 2.13. As with the earlier corollary, our proof reduces to showing

$$\left(\int_{\mathbf{R}^d}|Tg|^{p'}\,\sigma\,dx\right)^{1/p'}\le\left(\int_{\mathbf{R}^{d+1}_+}|g(t,y)|^{q'}\,d\mu(t,y)\right)^{1/q'},$$

for g bounded and with compact support in $\mathbf{R}^{d+1}_+$, where

$$Tg(x)=\int_{\mathbf{R}^{d+1}_+}g(t,y)\,y^{-1}\psi_y(t-x)\,d\mu(t,y).$$

Proceeding as we did there, we also write:

$$\begin{aligned}Tg(x)&=\sum_I\int_{T(I)}g(t,y)\,y^{-1}\psi_y(t-x)\,d\mu(t,y)\\&=\sum_I\lambda_I\phi_{(I)}(x),\end{aligned}$$

where each $\phi_{(I)}$ is as in the (2.1), and the λ_I's satisfy

$$|\lambda_I|\le c\left(\int_{T(I)}|g|^{q'}\,d\mu\right)^{1/q'}\mu(T(I))^{1/q}\ell(I)^{-(1+d/2)}.$$

By virtue of Theorem 2.10 and our hypothesis on σ and w (recall that $1<p'<2$),

$$\int_{\mathbf{R}^d}|Tg|^{p'}\,\sigma\,dx\le C\int_{\mathbf{R}^d}\left(\sum_I\frac{|\lambda_I|^2}{|I|}\sum_{j=0}^{\infty}3^{-2Mj+jd+j\epsilon}\chi_{3^jI}\right)^{p'/2}w\,dx.$$

Since $p'/2\le 1$, this last quantity is less than or equal to

$$\begin{aligned}C\int_{\mathbf{R}^d}\sum_{I,j}\left[\frac{|\lambda_I|^2}{|I|}\right]^{p'/2}3^{(-2Mj+jd+\epsilon)p'/2}\chi_{3^jI}\,w\,dx&\le C\sum_I\left[\frac{|\lambda_I|^2}{|I|}\right]^{p'/2}\int_{\mathbf{R}^d}\sum_j 3^{(-2Mj+jd+j\epsilon)p'/2}\chi_{3^jI}\,w\,dx\\&\le C\sum_I\left[\frac{|\lambda_I|^2}{|I|}\right]^{p'/2}\int_{\mathbf{R}^d}\frac{w(x)}{(1+|x-x_I|/\ell(I))^{(p'M-(p'/2)(d+\epsilon)}}.\end{aligned}$$

The proof now concludes as with the previous corollary. QED.

3. Two parameters.

Somewhat surprisingly, the method of Theorem 2.2 carries over directly to handle analogous two-parameter sums, at least for $L^2\mapsto L^2$. This generalization is possible because of a result from [W2], which gives a good Littlewood-Paley estimate for linear sums of (two-parameter) adapted functions. These are analogous to the adapted functions discussed above, but they satisfy some extra smoothness and cancellation conditions.

Before stating the result from [W2], and giving the requisite definitions, we shall first describe in detail the two-parameter problem we want to apply it to.

We are working on $\mathbf{R}^{d_1} \times \mathbf{R}^{d_2}$, which we think of as consisting of ordered pairs (x, y), with $x \in \mathbf{R}^{d_1}$ and $y \in \mathbf{R}^{d_2}$. We let $\mathcal{D}_i$ denote the family of dyadic cubes in $\mathbf{R}^{d_i}$. We shall not be concerned with functions which are, like the $\phi_{(I)}$'s above, centered around the dyadic cubes in $\mathbf{R}^{d_1} \times \mathbf{R}^{d_2}$, but, rather, to functions that are, in a certain sense, centered on rectangles $R = I \times J$, where $I \in \mathcal{D}_1$ and $J \in \mathcal{D}_2$. We suppose that, for every such R, we have a function $\phi_{[R]}(x, y)$. This function satisfies, for all x and y,

$$
\begin{aligned}
|\phi_{[R]}(x,y)| &\leq |I|^{-1/2}(1+|x-x_I|/\ell(I))^{-M_1}|J|^{-1/2}(1+|y-y_J|/\ell(J))^{-M_2} \\
|\nabla_x\phi_{[R]}(x,y)| &\leq \ell(I)^{-1}|I|^{-1/2}(1+|x-x_I|/\ell(I))^{-M_1-1}|J|^{-1/2}(1+|y-y_J|/\ell(J))^{-M_2} \\
|\nabla_y\phi_{[R]}(x,y)| &\leq |I|^{-1/2}(1+|x-x_I|/\ell(I))^{-M_1}\ell(J)^{-1}|J|^{-1/2}(1+|y-y_J|/\ell(J))^{-M_2-1} \\
|\nabla_x\nabla_y\phi_{[R]}(x,y)| &\leq \ell(I)^{-1}|I|^{-1/2}(1+|x-x_I|/\ell(I))^{-M_1-1}\ell(J)^{-1}|J|^{-1/2}(1+|y-y_J|/\ell(J))^{-M_2-1}
\end{aligned}
$$

for some exponents $M_i > d_i$ which are *fixed*. Moreover, each $\phi_{[R]}$ satisfies $\int_{\mathbf{R}^{d_1}} \phi_{[R]}(x,y)\,dx = 0$ for all y and $\int_{\mathbf{R}^{d_2}} \phi_{[R]}(x,y)\,dy = 0$ for all x.

Note that the bounds satisfied by each $\phi_{[R]}(x,y) = \phi_{I\times J}(x,y)$ are also satisfied by tensor products of the form $\phi_{(I)}(x) \cdot \phi_{(J)}(y)$. However, the functions $\phi_{[R]}(x,y)$ need not be tensor products; and, in our application to Bergman spaces below, they will not be.

We consider finite linear sums of the form:

$$
f(x,y) = \sum_R \lambda_R \phi_{[R]}(x,y). \tag{3.1}
$$

In order to state our main result we shall need a slightly modified form of one of our definitions from section 2.

Definition 3.1. *Let $\eta > 0$. If $R = I \times J \subset \mathbf{R}^{d_1} \times \mathbf{R}^{d_2}$ is a rectangle, as described above, and $v \in L^1_{loc}(\mathbf{R}^{d_1} \times \mathbf{R}^{d_2})$ is non-negative, we set*

$$
\tilde{v}(R,\eta) = \int_R v(x,y)\log^{\eta}(e + v(x,y)/v_R)\,dx\,dy,
$$

where v_R is v's average over R.

Our main result in this section is the following:

Theorem 3.2. *Let $\{\phi_{[R]}\}_R$ be as described above. Let $\epsilon > 0$ satisfy $2M_i - d_i > \epsilon > 0$ for $i = 1, 2$. Let $\eta > 2$. There is a constant $C = C(M_1, M_2, \epsilon, \eta, d_1, d_2)$ so that, for every finite sum* (3.1),

$$
\int_{\mathbf{R}^{d_1}\times\mathbf{R}^{d_2}} |f|^2\, v\,dx\,dy \leq C \sum_R \frac{|\lambda_R|^2}{|R|} \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} 3^{-M_1 j_1 - M_2 j_2 + j_1 d_1 + j_2 d_2 + (j_1+j_2)\epsilon}\tilde{v}(3^{j_1} I \times 3^{j_2} J, \eta).
$$

In particular, if $v(x,y)$ is uniformly A_∞ in x and y, then

$$
\begin{aligned}
&\int_{\mathbf{R}^{d_1}\times\mathbf{R}^{d_2}} |f|^2\, v\,dx\,dy \\
&\leq C' \int_{\mathbf{R}^{d_1}\times\mathbf{R}^{d_2}} \left(\sum_{I,J} \frac{|\lambda_R|^2}{|I||J|}(1+|x-x_I|/\ell(I))^{-2M_1+d_1+\epsilon}(1+|y-y_J|/\ell(J))^{-2M_2+d_2+\epsilon} \right) v\,dx\,dy,
\end{aligned}
$$

for a constant C' which depends only on M_1, M_2, d_1, d_2, ϵ, and the A_∞ parameters of v.

The result we shall use from [W2]—to which we alluded at the beginning of this section—depends on a two-parameter version of the adapted functions we used in section 2.

Definition 3.3. *Let $R = Q_1 \times Q_2$, where the Q_i are cubes in $\mathbf{R}^{d_i}$ (not necessarily dyadic). We say that $a_{[R]}(x,y)$ is* adapted *to R if $a_{[R]}$ is smooth and satisfies:*

i) supp $a_{[R]} \subset R$;
ii) $\int a_{[R]}(x,y)\,dx = 0$ for all $y \in Q_2$;
iii) $\int a_{[R]}(x,y)\,dy = 0$ for all $x \in Q_1$;
iv) $\|\nabla_x \nabla_y a_{[R]}\|_\infty \le \ell(Q_1)^{-1}\ell(Q_2)^{-1}|R|^{-1/2}$.

Here is the result we need from [W2] (Theorem 2.2 in that paper; also see the remark on page 434, after the end of the proof).

Theorem 3.4. *For $i = 1, 2$, let $\mathcal{G}_i$ be a good family of cubes in $\mathbf{R}^{d_i}$. Set $\tilde{\mathcal{G}} = \{R = Q_1 \times Q_2 :\ Q_i \in \mathcal{G}_i\}$. Let $\eta > 2$. There is a constant $C = C(d_1, d_2, \eta)$ such that, if*

$$f(x,y) = \sum_{R \in \tilde{\mathcal{G}}} \lambda_R a_{[R]}(x,y)$$

is any finite linear sum, where each $a_{[R]}$ is adapted to R, and v is any non-negative weight defined on $\mathbf{R}^{d_1} \times \mathbf{R}^{d_2}$, then

$$\int |f|^2\, v\, dx\, dy \le C \sum_R \frac{|\lambda_R|^2}{|R|} \tilde{v}(R,\eta).$$

The constant C does not depend on the family $\tilde{\mathcal{G}}$.

Proof of Theorem 3.2. We note that the "in particular" (see the statement of the theorem) follows immediately from the first conclusion; since, if v is uniformly A_∞ in both variables, then $\tilde{v}(R,\eta) \le C(v,\eta)\int_R v$ for all rectangles R. The result then follows from interchanging the order of summation and integration, much as in the proof of Theorem 2.2.

We begin by applying Lemma 2.3 to each $\phi_{[R]}(x,y)$ separately in the x and y variables. If we keep y fixed, we may write

$$\phi_{[R]}(x,y) = C(M_1, d_1) \sum_{j_1=1}^{\infty} 3^{-M_1 j_1} \phi_{[R],j_1}(x,y),$$

where each $\phi_{[R],j_1}(x,y)$ has x-support contained in $3^{j_1} I$ (recall that $R = I \times J$), has zero integral along each x and y slice, and satisfies:

$$\begin{aligned}
|\phi_{[R],j_1}(x,y)| &\le |I|^{-1/2}|J|^{-1/2}(1 + |y - y_J|/\ell(J))^{-M_2-1}\\
|\nabla_x \phi_{[R],j_1}(x,y)| &\le (3^{j_1}\ell(I))^{-1}|I|^{-1/2}|J|^{-1/2}(1 + |y - y_J|/\ell(J))^{-M_2-1}\\
|\nabla_y \phi_{[R],j_1}(x,y)| &\le |I|^{-1/2}\ell(J)^{-1}|J|^{-1/2}(1 + |y - y_J|/\ell(J))^{-M_2-1}\\
|\nabla_x \nabla_y \phi_{[R],j_1}(x,y)| &\le (3^{j_1}\ell(I))^{-1}|I|^{-1/2}\ell(J)^{-1}|J|^{-1/2}(1 + |y - y_J|/\ell(J))^{-M_2-1}
\end{aligned}$$

for all x and y. These estimates hold because, as is evident from Uchiyama's proof of Lemma 2.3, the functions $\phi_{(I),j}$ depend *smoothly* on the function $\phi_{(I)}$; for the sake of completeness, we have included a justification of this statement in an appendix. Now we may apply Lemma 2.3 to each $\phi_{[R],j_1}$ (in y, keeping x fixed this time), and obtain, after we put everything back together:

$$\phi_{[R]}(x,y) = C(M_1, d_1, M_2, d_2) \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} 3^{-M_1 j_1} 3^{-M_2 j_2} \phi_{[R],j_1,j_2}(x,y),$$

where each $\phi_{[R],j_1,j_2}(x,y)$ has support contained in $3^{j_1} I \times 3^{j_2} J$, has integral zero along its x and y slices, and satisfies:

$$\begin{aligned}
|\phi_{[R],j_1,j_2}(x,y)| &\le |R|^{-1/2}\\
|\nabla_x \phi_{[R],j_1,j_2}(x,y)| &\le (3^{j_1}\ell(I))^{-1}|R|^{-1/2}\\
|\nabla_y \phi_{[R],j_1,j_2}(x,y)| &\le (3^{j_2}\ell(J))^{-1}|R|^{-1/2}\\
|\nabla_x \nabla_y \phi_{[R],j_1,j_2}(x,y)| &\le (3^{j_1}\ell(I))^{-1}(3^{j_2}\ell(J))^{-1}|R|^{-1/2}
\end{aligned}$$

for all x and y.

Thus we may rewrite (3.1):

$$f(x,y) = C(M_1, M_2, d_1, d_2) \sum_{j_1, j_2} 3^{-M_1 j_1 - M_2 j_2} \sum_R \lambda_R \phi_{[R], j_1, j_2}(x,y).$$

We now apply Lemma 2.5 in x and y separately. The collection of cubes of the form $3^{j_1} I$ are divided into $3^{j_1 d_1}$ pairwise disjoint, good families $\mathcal{G}^1_{k_1}$ $(1 \leq k_1 \leq 3^{j_1 d_1})$, and the cubes of the form $3^{j_2} J$ are divided into $3^{j_2 d_2}$ pairwise disjoint, good families $\mathcal{G}^2_{k_2}$ $(1 \leq k_2 \leq 3^{j_2 d_2})$. If R is a rectangle of the form $R = 3^{j_1} I \times 3^{j_2} J$, where $3^{j_1} I \in \mathcal{G}^1_{k_1}$ and $3^{j_2} J \in \mathcal{G}^2_{k_2}$, we will assign R to the family $\tilde{\mathcal{G}}_{k_1,k_2}$. Note that the families $\tilde{\mathcal{G}}_{k_1,k_2}$ are pairwise disjoint and there are $3^{j_1 d_1 + j_2 d_2}$ of them.

We rewrite our sum as:

$$C(M_1, M_2, d_1, d_2) \sum_{j_1, j_2} \sum_{\substack{k_1 = 1 \ldots 3^{j_1 d_1} \\ k_2 = 1 \ldots 3^{j_2 d_2}}} \sum_{3^{j_1} I \times 3^{j_2} J \in \tilde{\mathcal{G}}_{k_1,k_2}} 3^{-M_1 j_1 - M_2 j_2} \lambda_R \phi_{[R], j_1, j_2}(x,y).$$

Following the notational convention we used in the proof of Theorem 2.2, we rewrite this last quantity as:

$$C(M_1, M_2, d_1, d_2) \sum_{j_1, j_2} \sum_{\substack{k_1 = 1 \ldots 3^{j_1 d_1} \\ k_2 = 1 \ldots 3^{j_2 d_2}}} \sum_{R = 3^{j_1} I \times 3^{j_2} J \in \tilde{\mathcal{G}}_{k_1,k_2}} 3^{-M_1 j_1 - M_2 j_2} 3^{(j_1 d_1 + j_2 d_2)/2} \lambda_R \tilde{\phi}_{[R], j_1, j_2}(x,y),$$

where now each $\tilde{\phi}_{[R], j_1, j_2}$ is *adapted* (in the "rectangle") sense to $3^{j_1} I \times 3^{j_2} J$. Now we shall use the Theorem 3.4, cited earlier. Each of the sums

$$\sum_{R = 3^{j_1} I \times 3^{j_2} J \in \tilde{\mathcal{G}}_{k_1,k_2}} 3^{-M_1 j_1 - M_2 j_2} 3^{(j_1 d_1 + j_2 d_2)/2} \lambda_R \tilde{\phi}_{[R], j_1, j_2}(x,y)$$

has the form

$$\sum_{\substack{R^* = I^* \times J^* \\ I^* \in \mathcal{G}^*_1, J^* \in \mathcal{G}^*_2}} \gamma_{R^*} \tilde{\phi}_{[R^*]}(x,y),$$

where the families $\mathcal{G}^*_1$ and $\mathcal{G}^*_2$ are good, each $\tilde{\phi}_{[R^*]}$ is adapted to $R^* = 3^{j_1} I \times 3^{j_2} J$, and we have set $\gamma_{R^*} = 3^{-M_1 j_1 - M_2 j_2} 3^{(j_1 d_1 + j_2 d_2)/2} \lambda_R$. Therefore, for every $\eta > 2$, there is a constant $C = C(\eta, d_1, d_2)$ such that

$$\int_{\mathbf{R}^{d_1} \times \mathbf{R}^{d_2}} \Big| \sum_{\substack{R^* = I^* \times J^* \\ I^* \in \mathcal{G}^*_1, J^* \in \mathcal{G}^*_2}} \gamma_{R^*} \tilde{\phi}_{[R^*]}(x,y) \Big|^2 v \, dx \, dy \leq C \sum_{R^* = I^* \times J^*} \frac{|\gamma_{R^*}|^2}{|I^*||J^*|} \tilde{v}(I^* \times J^*, \eta),$$

valid for any non-negative weight v. Applied to our original sum, this yields:

$$\int_{\mathbf{R}^{d_1} \times \mathbf{R}^{d_2}} \Big| \sum_{\substack{R = I \times J \\ 3^{j_1} I \times 3^{j_2} J \in \tilde{\mathcal{G}}_{k_1,k_2}}} 3^{-M_1 j_1 - M_2 j_2} 3^{(j_1 d_1 + j_2 d_2)/2} \lambda_R \tilde{\phi}_{[R], j_1, j_2}(x,y) \Big|^2 v \, dx \, dy$$

$$\leq C(\eta, d_1, d_2) \sum_{\substack{R = I \times J \\ 3^{j_1} I \times 3^{j_2} J \in \tilde{\mathcal{G}}_{k_1,k_2}}} 3^{-2M_1 j_1 - 2M_2 j_2} \frac{|\lambda_R|^2}{|I||J|} \tilde{v}(3^{j_1} I \times 3^{j_2} J, \eta),$$

for each ordered pair (j_1, j_2), and when $1 \leq k_1 \leq 3^{j_1 d_1}$ and $1 \leq k_2 \leq 3^{j_2 d_2}$. The rest of the proof now follows from two applications of the Cauchy-Schwarz inequality, exactly as in the proof of Theorem 2.2. QED.

Theorem 3.2 leads directly to a two-parameter generalization of the corollary from the previous section, at least when $1 < p \leq 2 \leq q < \infty$.

For the rest of the paper, in order to make the statements of our results a little more compact, and to adhere to the tradition which says that y must always represent a dilation parameter, we will be changing our notation somewhat. Henceforth, points in $\mathbf{R}^{d_1} \times \mathbf{R}^{d_2}$ will be denoted by (x_1, x_2), and y_1 and y_2 will denote postive numbers. Thus:

$$\mathbf{R}_+^{d_1+1} \times \mathbf{R}_+^{d_2+1} = \{(x_1, y_1, x_2, y_2) : \; x_i \in \mathbf{R}^{d_i}, \, y_i > 0\}.$$

Corollary 3.5. *Let M_1, M_2, d_1, d_2, and ϵ be as in Theorem 3.2. For $i = 1$, 2, let ϕ_i be smooth functions defined on $\mathbf{R}^{d_i}$, and satisfying (for $x_i \in \mathbf{R}^{d_i}$),*

$$\begin{aligned} |\phi_i(x_i)| &\le (1 + |x_i|)^{-M_i} \\ |\nabla \phi_i(x_i)| &\le (1 + |x_i|)^{-M_i-1} \\ \int_{\mathbf{R}^{d_i}} \phi_i \, dx_i &= 0. \end{aligned}$$

For $y_i > 0$, define $y = (y_1, y_2)$ and set $\Phi_y(x_1, x_2) = (\phi_1)_{y_1}(x_1) \cdot (\phi_2)_{y_2}(x_2)$. Consider the two-parameter Bergman-type inequality:

$$\left(\int_{\mathbf{R}_+^{d_1+1} \times \mathbf{R}_+^{d_2+1}} |y_1^{-1} y_2^{-1} f * \Phi_y(x_1, x_2)|^q \, d\mu(x_1, x_2, y) \right)^{1/q} \le \left(\int_{\mathbf{R}^{d_1} \times \mathbf{R}^{d_2}} |f|^p \, v \, dx_1 \, dx_2 \right)^{1/q}, \tag{3.2}$$

where μ is a Borel measure defined on $\mathbf{R}_+^{d_1+1} \times \mathbf{R}_+^{d_2+1}$, f is assumed to belong to a reasonable test class, and v is a non-negative, locally-integrable function defined on $\mathbf{R}^{d_1} \times \mathbf{R}^{d_2}$. Let $1 < p \le 2 \le q < \infty$ and set $\sigma = v^{1-p'}$. In order that (3.2) should hold for all f, it is sufficient that there exist a number $\tau > p'$, an $\epsilon > 0$, and a weight w such that

$$\tilde{\sigma}(R, \tau) \le \int_R w$$

and

$$\mu(T(I) \times T(J))^{1/q} \left(\int_{\mathbf{R}^{d_1} \times \mathbf{R}^{d_2}} \frac{w(x_1, x_2)}{(1 + |x_1 - x_I|/\ell(I))^{p' M_1 - (p'/2)(d_1+\epsilon)} (1 + |x_2 - x_J|/\ell(J))^{p' M_2 - (p'/2)(d_2+\epsilon)}} \, dx_1 \, dx_2 \right)^{1/p'}$$
$$\le c \ell(I)^{d_1+1} \ell(J)^{d_2+1}$$

for all rectangles $R = I \times J$.

Proof of Corollary 3.5. We consider the dual form of (3.2):

$$\left(\int_{\mathbf{R}^{d_1} \times \mathbf{R}^{d_2}} |\tilde{T} g|^{p'} \, \sigma \, dx_1 \, dx_2 \right)^{1/p'} \le \left(\int_{\mathbf{R}_+^{d_1+1} \times \mathbf{R}_+^{d_2+1}} |g(t_1, t_2, y)|^{q'} \, d\mu(t_1, t_2, y) \right)^{1/q'},$$

where g is bounded and has compact support in $\mathbf{R}_+^{d_1+1} \times \mathbf{R}_+^{d_2+1}$; and

$$\tilde{T} g(x_1, x_2) = \int_{\mathbf{R}_+^{d_1+1} \times \mathbf{R}_+^{d_2+1}} g(t_1, t_2, y) \, y_1^{-1} y_2^{-1} \Phi_y(t_1 - x_1, t_2 - x_2) \, d\mu(t_1, t_2, y).$$

Set $x = (x_1, x_2)$, $t = (t_1, t_2)$, and write

$$\begin{aligned} \tilde{T} g(x) &= \sum_R \int_{T(I) \times T(J)} g(t, y) \, y_1^{-1} y_2^{-1} \Phi_y(t - x) \, d\mu(t, y) \\ &= \sum_R \lambda_R \phi_{[R]}(x, y), \end{aligned} \tag{3.3}$$

where $T(I)$ and $T(J)$'s are the usual top halves of Carleson boxes, each $\phi_{[R]}$ satisfies the hypotheses of Theorem 3.2, and the λ_R's satisfy

$$|\lambda_R| \leq c \left(\int_{T(I)\times T(J)} |g|^{q'} \, d\mu \right)^{1/q'} \mu(T(I)\times T(J))^{1/q} \ell(I)^{-(1+d_1/2)} \ell(J)^{-(1+d_2/2)}.$$

Note that, because of g's compact support, the sum (3.3) is finite.

Let ρ be the dual exponent to $p'/2$, and let h be non-negative and satisfy $\int |h|^\rho \, \sigma = 1$. We need to show that

$$\int |\tilde{T}g|^2 \, h \, \sigma \, dx_1 \, dx_2 \leq \left(\int_{\mathbf{R}_+^{d_1+1}\times\mathbf{R}_+^{d_2+1}} |g(t_1,t_2,y)|^{q'} \, d\mu(t_1,t_2,y) \right)^{2/q'},$$

independent of the particular choice of h. We now apply Main Theorem 3, setting $v = h\sigma$. We get:

$$\int |\tilde{T}g|^2 \, h \, \sigma \, dx_1 \, dx_2 \leq C \sum_R \frac{|\lambda_R|^2}{|R|} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} 3^{-2M_1 j_1 - 2M_2 j_2 + j_1 d_1 + j_2 d_2 + (j_1+j_2)\epsilon} \widetilde{(h\sigma)}(3^{j_1} I \times 3^{j_2} J, \eta),$$

where we have chosen $\eta = 2\tau/p' > 2$ (i.e., $\tau = \eta p'/2$). The argument from [W3] shows that, for any rectangle R,

$$\widetilde{(h\sigma)}(R) \leq C(d_1,d_2,p) \tilde{\sigma}(R, \eta p'/2)^{2/p'}.$$

But this last quantity is less than or equal to $Cw(R)^{2/p'}$. Thus, we may dominate our sum by

$$C \sum_{R=I\times J} \frac{|\lambda_R|^2}{|R|} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} 3^{-2M_1 j_1 - 2M_2 j_2 + j_1 d_1 + j_2 d_2 + (j_1+j_2)\epsilon} w(3^{j_1} I \times 3^{j_2} J)^{2/p'}.$$

At the cost of slightly increasing ϵ, we may dominate

$$\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} 3^{-2M_1 j_1 - 2M_2 j_2 + j_1 d_1 + j_2 d_2 + (j_1+j_2)\epsilon} w(3^{j_1} I \times 3^{j_2} J)^{2/p'}$$

by

$$C_\epsilon \left(\int_{\mathbf{R}^{d_1}\times\mathbf{R}^{d_2}} \frac{w(x_1,x_2)}{[(1+|x_1-(x_1)_I|/\ell(I))^{2M_1-(d_1+\epsilon)}(1+|x_2-(x_2)_J|/\ell(J))^{2M_2-(d_2+\epsilon)}]^{p'/2}} dx_1 \, dx_2 \right)^{2/p'}.$$

(Note: here we are following, almost verbatim, the procedure of Corollary 2.8; see above.) Therefore,

$$\int |\tilde{T}g|^2 \, h \, \sigma \, dx_1 \, dx_2 \leq$$

$$C \sum_{R=I\times J} \frac{|\lambda_R|^2}{|R|} \left(\int_{\mathbf{R}^{d_1}\times\mathbf{R}^{d_2}} \frac{w(x)}{[(1+|x_1-(x_1)_I|/\ell(I))^{2M_1-(d_1+\epsilon)}(1+|x_2-(x_2)_J|/\ell(J))^{2M_2-(d_2+\epsilon)}]^{p'/2}} dx_1 \, dx_2 \right)^{2/p'},$$

where the λ_R's have the bounds given above. The rest of the proof, is now *exactly* like that from Corollary 2.8, and we leave it to the interested reader. QED.

Final Remarks. In [WW1], Richard Wheeden and the author proved sufficient conditions, on weights v and measures μ, for the Bergman-type inequality

$$\left(\int_{\mathbf{R}_+^{d+1}} |y^{-1}\psi_y * f(x)|^q \, d\mu(x,y) \right)^{1/q} \leq \left(\int_{\mathbf{R}^d} |f|^p \, v \, dx \right)^{1/p}$$

to hold for all f in a reasonable test class, and for certain smooth convolution kernels ψ that satisfied $\int \psi = 0$. It may be instructive to compare the results obtained in [WW1] to those proved here. In most respects, but not all, those from the present paper are stronger.

1) The general result obtained in [WW1] required that the smooth kernel ψ satisfy:

$$|\psi(x)| \leq (1+|x|)^{-M}$$
$$\ell(I)|\nabla\psi(x)| \leq (1+|x|)^{-M-1},$$

for all $x \in \mathbf{R}^d$, *for some* $M \geq d+2$. The methods of this paper only require that $M > d$. So, in this respect, the results from the present paper are stronger.

On the other hand:

2) In both [WW1] and the present paper, the sufficient condition obtained was (essentially) of the form:

$$\mu(T(I))^{1/q}\left(\int_{\mathbf{R}^d} \frac{w(x)}{(1+|x-x_I|/\ell(I))^\rho}\right)^{1/p'} \leq C\ell(I)^\nu,$$

for all cubes $I \subset \mathbf{R}^d$, for certain exponents ρ and ν, depending (possibly) on p, q, and d; we say "essentially" because the result from [WW1] also included a power of a logarithm in the numerator of the integrand. The exponent to watch here is ρ: the bigger ρ is (i.e., the more decay in the w-integral), the better the theorem. In [WW1], $\rho = p'/q'$; here, $\rho = p'M - (p'/2)(d+\epsilon)$. Thus, the result from the present paper is better when $M > qd/2$, but [WW1]'s is better when $M < qd/2$ (unless $d < M < d+2$, where [WW1] gave no general result). The author does not know yet what to make of this puzzling phenomenon.

Remark. Just as this paper was being accepted for publication, the author learned of an argument which improves the value of $\tilde{c}(p)$ (see Definition 2.9) in Theorem 2.10, for p *outside* the range of $1 \leq p \leq 2$. Unfortunately, this argument offers no improvement for $1 < p \leq 2$, which is the range needed for our main application (Bergman space inequalities), nor is it clear at this time that it can be extended to the two-parameter setting.

Appendix: Justification of continuity.

We refer the reader to the proof of Lemma 3.5 in [U].

By dilation and translation, we may assume that we have a function smooth $\phi(x,y)$, defined on $\mathbf{R}^{d_1} \times \mathbf{R}^{d_2}$, satisfying:

$$|\phi(x,y)| \leq (1+|x|)^{-M_1}(1+|y|)^{-M_2}$$
$$|\nabla_x\phi(x,y)| \leq (1+|x|)^{-M_1-1}(1+|y|)^{-M_2}$$
$$|\nabla_y\phi(x,y)| \leq (1+|x|)^{-M_1}(1+|y|)^{-M_2-1}$$
$$|\nabla_x\nabla_y\phi(x,y)| \leq (1+|x|)^{-M_1-1}(1+|y|)^{-M_2-1}$$

for some $M_i > d_i$. We furthermore assume that ϕ has integral zero along each of its x and y slices.

Let $h \in \mathcal{C}^\infty(\mathbf{R})$ be a non-negative function with support contained in $(.1, .9)$, and which satisfies

$$\sum_{j=1}^{\infty} h(3^{-j}t) \equiv 1$$

for $t > 1$. Set

$$h_0(t) = 1 - \sum_{j=1}^{\infty} h(3^{-j}t).$$

Then, for any $y \in \mathbf{R}^{d_2}$,

$$\begin{aligned}
\phi(x,y) &= h_0(|x|)\phi(x,y) + \sum_{j=1}^{\infty} h(3^{-j}|x|)\phi(x,y) \\
&= \left[h_0(|x|)\phi(x,y) + h(|x|)\frac{\int_{\mathbf{R}^{d_1}} \sum_{k=1}^{\infty} h(3^{-k}|t|)\phi(t,y)\,dt}{\int_{\mathbf{R}^{d_1}} h(|t|)\,dt} \right] \\
&+ \sum_{j=1}^{\infty} [h(3^{-j}|x|)\phi(x,y) - h(3^{-j+1}|x|)\frac{\int_{\mathbf{R}^{d_1}} \sum_{k=j}^{\infty} h(3^{-k}|t|)\phi(t,y)\,dt}{\int_{\mathbf{R}^{d_1}} h(3^{-j+1}|t|)\,dt} \\
&+ h(3^{-j}|x|)\frac{\int_{\mathbf{R}^{d_1}} \sum_{k=j+1}^{\infty} h(3^{-k}|t|)\phi(t,y)\,dt}{\int_{\mathbf{R}^{d_1}} h(3^{-j}|t|)\,dt}] \\
&= \tilde{\beta}_0(x,y) + \sum_{j=1}^{\infty} \tilde{\beta}_j(x,y).
\end{aligned}$$

The $\tilde{\beta}_j$'s smoothness in x and cancellation (in x), as well as the support properties, follow as they do in Uchiyama's proof. Cancellation in y is trivial. What is not so trivial, but also not hard, is the *smoothness* in y of the $\tilde{\beta}_j$'s. However, it is easy to see (by, say, the Dominated Convergence Theorem) that

$$\begin{aligned}
|\nabla_y \tilde{\beta}_j(x,y)| &\le C(M_1,d_1)3^{-jd_1} \int_{|t|>c3^j} |\nabla_y \phi(t,y)|\,dt \\
&\le C3^{-jd_1}(1+|y|)^{-M_2-1}3^{jd_1-jM_1} \\
&= C3^{-M_1 d_1}(1+|y|)^{-M_2-1},
\end{aligned}$$

with a corresponding estimate for $\nabla_x \nabla_y \tilde{\beta}_j$. This justifies our statement about "continuity in ϕ."

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