Weighted Inequalities for Caloric Functions on Classical Domains

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Abstract: - Conditions on non-negative measures $d\mu(\vec{x},t)$ and $v(x',t)\,d\sigma(x',t)$ are given and shown to be sufficient to have the inequality

$$\left(\int_{\Omega}|\nabla u(\vec{x},t)|^{q}\,d\mu(\vec{x},t)\right)^{\frac{1}{q}} \le C\left(\int_{\partial_p\Omega}|f(x',t)|^{p}\,v(x',t)\;d\sigma(x',t)\right)^{\frac{1}{p}}$$

hold for appropriate p and q, where $Lu = 0$ in Ω and $u\mid_{\partial_p\Omega}= f$, on several different domains Ω. The method of proof is that of Wilson and Wheeden [4] to use the dual operator and to establish a Littlewood-Paley inequality for families of functions $\sum\lambda_{(I)}\varphi_{(I)}$ where the $\{\varphi_{(I)}\}$ are adapted to dyadic parabolic cubes I. The operator $L=\frac{\partial}{\partial t}-\Delta$, i.e. L is the heat operator on the right half space and on the semi-infinite box domain for $\sigma(x',t)$ being Lebesgue measure.

Key-Words: - Heat equation, gradient estimates, weights, discrete decompositions, Littlewood-Paley inequalities.

The Heat Equation.

In dealing with solutions to the heat equation it is useful to be able to estimate the gradient of a solution. Local estimates for averages of the gradient of a parabolic function are frequently given by energy inequalities or p-Cacciopoli inequalities. However, it is also important to establish global estimates on ∇u when u is a weak solution to the Dirichlet problem:

$$\left(\frac{\partial}{\partial t}-\Delta\right)u = 0 \; in\,\Omega, u\mid_{\partial_p\Omega}= f\,.$$

Such Bergman space estimates easily give sufficient conditions so that $\|Su\|_{L^q(\partial\Omega)} \le c\|f\|_{L^q(\partial\Omega)}$ where Su is the Lusin area integral. In addition one would like to know conditions on measures to have $\|Su\|_{L^q(d\mu,\partial\Omega)} \le c\|f\|_{L^q(dv,\partial\Omega)}$. This can be easily done by establishing sufficient conditions for μ and v so that

$$\left(\int_{\Omega}|\nabla u(x,t)|^{q}\,d\mu(x,t)\right)^{\frac{1}{q}} \le C\left(\int_{\partial_p\Omega}|f(x',t)|^{p}\,dv(x',t)\right)^{\frac{1}{p}}. \quad (*)$$

Using the dual operator, as Wilson and Wheeden do in [4], one can establish conditions on μ and v so the above inequality holds for a wide range of exponents p and q. The key estimate that allows the proof to work is to prove a Littlewood-Paley inequality for finite sums of the form

$\sum \lambda_{(I)}\varphi_{(I)}(x',t)$, where $\{\varphi_{(I)}\}_{I\in\mathfrak{G}}$ are a family of functions with certain decay properties. The fact that solutions to the heat equation can be dealt with by a partial discretization process is of major importance.

We start with the situation where the operator is the heat operator $\frac{\partial}{\partial t}-\Delta$ on Ω = right half plane. The box domain, $R=\{(\vec{x},t): 0<x_i<1, i=1,2,\ldots,d,-\infty<t<T\}$ will be shown to have similar estimates hold for the same kind of conditions on $\mu, v(x',t)\,dx'dt$ and p and q for the heat equation. In these two domains the weakest conditions are enough to give

$$\left(\int_{\Omega}|\nabla u(x,t)|^q\,d\mu(x,t)\right)^{\frac{1}{q}}$$

$$\leq C\left(\int_{\partial_p\Omega}|f(x',t)|^p\,v(x',t)\,dx'dt\right)^{\frac{1}{p}};$$

for example $v(x',t)$ does not need an A^∞ assumption. Of course it is not surprising that this happens since the heat kernel has excellent decay properties for these two domains. We can prove the inequality (*) for $\nabla_t u$ as well as for $\nabla_x u$.

The Littlewood-Paley inequality that appears here, for the heat operator on $\mathbb{R}_r^{d+1}$ and R = the box domain, is a parabolic version of Theorem 2.2 in [6]. Theorem 1 below gives the parabolic version of Theorem 2.2 [6].

The heat equation on two classical domains.

Before stating and proving Theorem I we establish some notation:

The cubes, I, will be parabolic dyadic cubes in $\mathbb{R}^d=\mathbb{R}^{d-1}\times\mathbb{R}^1$, so a typical cube is

$$I=\left[\frac{j_1}{2^k},\frac{j_1+1}{2^k}\right]\times\left[\frac{j_2}{2^k},\frac{j_2+1}{2^k}\right]\times\ldots$$
$$\times\left[\frac{j_{d-1}}{2^k},\frac{j_{d-1}+1}{2^k}\right]\times\left[\frac{j_t}{4^k},\frac{j_t+1}{4^k}\right]$$

where j_i are integers. The collection of all such cubes $-\infty<k<\infty$ will be denoted by $\mathfrak{DP}$.

Let $\{\varphi_{(I)}\}_{I\in\mathfrak{DP}}$ be a family of functions indexed by parabolic dyadic cubes, so that

(1) $\int \varphi_{(I)}(x',t)\;dx'\,dt=0$

(2) $\left|\varphi_{(I)}(x',t)\right|\leq\frac{1}{\sqrt{|I|}}\left(1+\frac{d_p\left(x',t;x'_{(I)},t_{(I)}\right)}{\ell(I)}\right)^{-M}$

(3) $\ell(I)\left|\nabla_x\varphi_{(I)}(x',t)\right|\leq\frac{1}{\sqrt{|I|}}$

$$\left(1+\frac{d_p\left(x',t;x'_{(I)},t_{(I)}\right)}{\ell(I)}\right)^{-M-1}$$

$$\ell(I)^2\left|\nabla_t\varphi_{(I)}(x',t)\right|\leq\frac{1}{\sqrt{|I|}}\left(1+\frac{d_p\left(x',t;x'_{(I)},t_{(I)}\right)}{\ell(I)}\right)^{-M-2}$$

Here $d_p(x',t;y',s)=|x'-y'|+|t-s|^{\frac{1}{2}}$ and $d_p(\vec{x},t;\vec{y},s)=|\vec{x}-\vec{y}|+|t-s|^{\frac{1}{2}}$ where $\vec{x}=(x',x_d)$ $x'\in\mathbb{R}^{d-1}$ and $t\in\mathbb{R}^1$, $\ell(I)=2^{-k}$ $\ell(I)^2$ denotes the time dimension of I which is 4^{-k} if I is as above, so of course $|I|=\ell(I)^{d+1}$ is the Lebesgue measure of I. $(x'_{(I)},t_{(I)})$ is the center of I.

The following version of Theorem 2.2. in "A discrete Littlewood Paley inequality" is valid:

Theorem I. *For* $\{\varphi_{(I)}\}_{I\in\mathfrak{DP}}$ *satisfying (1) – (3) and M > d + 1, suppose* $f(x',t)=\sum_{I\in\mathfrak{F}}\lambda_{(I)}\varphi_{(I)}\;(x',t)$ *where* $\mathfrak{F}$ *is any finite family of parabolic dyadic cubes. For* $v\geq 0$, $v\in L^1_{loc}(\mathbb{R}^d)$ *there is a constant c = c(d,M,ϵ,η) so that*

$$\int_{\mathbb{R}^d}|f(x',t)|^2\,v(x',t)\,dx'dt\leq$$

$$c\sum_{I\in\mathfrak{F}}\frac{|\lambda_I|^2}{|I|}\sum_{j=0}^{\infty}3^{-j(2M-d-1-\epsilon)}\,v(3^jI;\eta)$$

where

$$v(I,\eta)=\int_I v(x',t)\log^{\eta}\left(e+\frac{v(x',t)}{v_I}\right)dx'dt$$

and $\eta > 1$,

$$v_i = \fint_I v(x',t)\,dx'dt = \frac{1}{|I|}\int_I v(x',t)\,dx'dt.$$

$3^j I$ is the 3^j-fold dilate of I, i.e. $\ell(3^j I)=3^j\ell(I)=$ space dimension of $3^j I$ and the time dimension is $9^j\ell(I)^2$. $3^j I$ is concentric with I.

Theorem 1 is a weighted, parabolic version of a discrete square function result of Uchiyama [3]. It was first proved for I being a Euclidean cube by J. M. Wilson. Although here we only use the parabolic version of Uchiyama's Lemma 3.5 (see Lemma 1 below) one might note that both Lemmas 3.5 and 3.3 in [3] have valid parabolic versions.

To prove Theorem 1 we proceed as in the proof of Theorem 2.2 [6] and prove parabolic versions of Lemmas 2.3, 2.5 and 2.7 in that paper.

Lemma 1. *For* $\varphi_{(I)}(x',t)$ *satisfying (1) – (3),* $\varphi_{(I)}(x',t)$ *can be written as a sum*

$$c(M,d)\sum_{j=0}^{\infty}3^{-Mj}\varphi_{(I),j}(x',t)$$

where

$$supp\ \varphi_{(I),j}\subseteq 3^j I$$

$$\left\|\varphi_{(I),j}\right\|_{\infty}\le\frac{1}{\sqrt{|I|}}$$

$$\left\|\varphi_{(I),j}\right\|_{Lip\left(1,\frac{1}{2}\right)}\le\left(\left(3^j\ell(I)\right)^{-1}\frac{1}{\sqrt{|I|}}\right)\ \textit{and}$$

$$\int_{\mathbb{R}^d}\varphi_{(I),j}(x',t)\,dx'\,dt=0.$$

Essentially Uchiyama's proof goes through with dyadic parabolic cubes replacing dyadic Euclidean cubes; as in [6] one can have any $M > d+1$ and 2^j is replaced by 3^j.

Proof. As Uchiyama does we may assume $(x'_I,t_I)=(0,0)$. Let $h(r)\in C^{\infty}(\mathbb{R}^1)$, $h\ge 0$, such that supp $h\subseteq\left(\frac{\ell(I)}{18},\frac{4\ell(I)}{9}\right)$ and so that

$$\sum_{j=1}^{\infty}h(3^{-j}r)=1 \text{ if } r>\frac{\ell(I)}{2} \text{ and}$$

$$h_0(r)=1-\sum_{j=1}^{\infty}h(3^{-j}r).$$

Then

$$\varphi_{(I)}(x',t)=h_0(\delta(x,t))\,\varphi_{(I)}(x',t)$$
$$+\sum_{j=1}^{\infty}h\left(3^{-j}\delta(x',t)\right)\cdot\varphi_{(I)}(x',t)$$

where $\delta(x',t)=d_p\left((x',t);(x'_{(I)},t_{(I)})\right)$ and $\varphi_{(I)}(x',t)$ can be written as a sum,

$$\sum_{j=0}^{\infty}\tilde{\beta}_j(x',t),\text{ where}$$

$$\tilde{\beta}_0(x',t)=h_0(\delta(x',t))\,\varphi_{(I)}(x',t)+\alpha_0 h(\delta(x',t))$$

and $\tilde{\beta}_j(x',t)=h\left(3^{-j}\delta(x',t)\right)\varphi_{(I)}(x',t)+$
$\alpha_j h\left(3^{-j}\delta(x',t)\right)-\alpha_{j-1}h\left(3^{-(j-1)}\delta(x',t)\right),$

if $j\ge 1$. For $j\ge 0$ we have

$$a_j=\frac{\int\sum_{k=j+1}^{\infty}h\left(3^{-k}\delta(y',s)\right)\varphi_{(I)}(y',s)\,dy'ds}{\int h\left(3^{-j}\delta(y',s)\right)dy'ds}.$$

The $\varphi_{(I),j}(x',t)$ in the Lemma are $3^{mj}\tilde{\beta}_j(x',t)$. To show that the properties described in Lemma 1 hold for $\varphi_{(I),j}$ first notice that $h_0(r)=0$ if $r>\frac{\ell(I)}{2}$ so supp $h_0\subseteq I$. Also $h(\delta(x',t))=0$ if $\delta(x',t)\ge\frac{4\ell(I)}{9}$ so supp $h\subseteq I$. This implies that supp $\tilde{\beta}_0\subseteq I$. For $j>0$,
supp $\tilde{\beta}_j\subseteq$ supp $h\left(3^{-j}\delta(x',t)\right)\cup$ supp $h\left(3^{-j+1}\delta(x',t)\right)$.
This means $3^{-j}\delta(x',t)<\frac{4\ell(I)}{9}$ or $\delta(x',t)<\frac{4}{3}\cdot 3^{j-1}\ell(I)<\frac{1}{2}\ell(3^j I)\Rightarrow x',t\in 3^j I$ so supp $\tilde{\beta}_j\subseteq 3^j I$ as required.

Now $\int\tilde{\beta}_0(x',t)\,dx'dt=0$ since $\int\varphi_{(I)}=0$ and $\int\tilde{\beta}_j(x',t)\,dx'dt=0$ also. To estimate

$$\left\|\varphi_{(I),j}\right\|_{\infty}\text{ and }\left\|\varphi_{(I),j}\right\|_{Lip\left(1,\frac{1}{2}\right)}$$

notice that

$$\left\|\varphi_{(I),0}\right\|_\infty \le \left\|\varphi_{(I)}\right\|_\infty \le \frac{1}{\sqrt{|I|}} \text{ and}$$

$$\left\|\varphi_{(I),0}\right\|_{Lip\left(1,\frac{1}{2}\right)} \le \left\|h_0\varphi_{(I)}\right\|_{Lip\left(1,\frac{1}{2}\right)} + |\alpha_0| \, \|h\|_{Lip\left(1,\frac{1}{2}\right)}$$

$$\le \left\|\varphi_{(I)}\right\|_{Lip\left(1,\frac{1}{2}\right)} + \left\|\sum_{j=1}^{\infty} h\left(3^{-j}\delta(x',t)\right)\varphi_{(I)}(x',t)\right\|_{Lip\left(1,\frac{1}{2}\right)}$$

$$+\frac{c}{\ell(I)}|\alpha_0| \le \frac{1}{\ell(I)\sqrt{|I|}} +$$

$$\left\|\sum_{j=1}^{\infty} h\left(3^{-j}\delta(x',t)\right)\varphi_{(I)}(x',t)\right\|_{Lip\left(1,\frac{1}{2}\right)}$$

$$+\frac{c}{\ell(I)}\frac{\frac{1}{\sqrt{|I|}}|I|}{|supp\ h|} \le \frac{c(d)}{\ell(I)\sqrt{|I|}}$$

from elementary facts about $h, \varphi_{(I)}$ and $Lip\left(1,\frac{1}{2}\right)$ norms.

For $j \ge 1$, first

$$|a_j| = \frac{\left|\int \sum_{k=j+1}^{\infty} h\left(3^{-k}\delta(y',s)\right)\cdot \varphi_{(I)}(y',s)\,dy'ds\right|}{\left|\int h\left(3^{-j}\delta(y',s)\right)dy'ds\right|}$$

$$\lesssim \sum_{k=j+1}^{\infty} \frac{\left|\int h\left(3^{-k}\delta(y',s)\right)\varphi_{(I)}(y',s)\,dy'ds\right|}{\left|\text{supp } h\left(3^{-j}\delta(y',s)\right)\right|}$$

$$\lesssim \sum_{k=j+1}^{\infty} \frac{\left|\text{supp } h\left(3^{-k}\ \cdot\right)\right| \left\|\varphi_{(I)}\right\|_{\infty,\text{supp } h\left(3^{-k}\cdot\right)}}{\left|3^{j-2}I\right|}$$

$$= \frac{1}{3^{(j-2)(d+1)}}\sum_{k=j+1}^{\infty} 3^{(k-1)(d+1)} \cdot$$

$$\frac{1}{\sqrt{|I|}}\left(1+\frac{d_p\left(\text{supp } h\left(3^{-k}\ \cdot\right);\vec{0}\right)}{\ell(I)}\right)^{-M}$$

$$\lesssim 3^{-j(d+1)}\frac{1}{\sqrt{|I|}}\sum_{k=j+1}^{\infty} 3^{k(d+1)}\cdot 3^{-kM} \approx \frac{3^{-jM}}{\sqrt{|I|}}$$

if $M > d + 1$. The constants depend on M and d.

Now

$$\left\|\tilde{\beta}_j\right\|_{Lip\left(1,\frac{1}{2}\right)} \le \left\|h\left(3^{-j}\delta(\cdot)\right)\varphi_{(I)}(\cdot)\right\|_{Lip\left(1,\frac{1}{2}\right)}$$

$$+\left|a_j\right| \left\|h\left(3^{-j}\delta(\cdot)\right)\right\|_{Lip\left(1,\frac{1}{2}\right)}$$

$$+\left|a_{j-1}\right| \left\|h\left(3^{-(j-1)}\delta(\cdot)\right)\right\|_{Lip\left(1,\frac{1}{2}\right)}$$

so using

$$\left\|h\left(3^{-j}\delta(\cdot)\right)\right\|_{Lip\left(1,\frac{1}{2}\right)} = \sup_{(x',t),(y',s)}$$

$$\left|\frac{h\left(3^{-j}\delta(x',t)\right)-h\left(3^{-j}\cdot\delta(y',s)\right)}{d_p(x',t;y',s)}\right| \le \|h'\|_\infty \cdot$$

$$\frac{3^{-j}\left(\delta(x',t)-\delta(y',s)\right)}{d_p(x',t;y',s)} \lesssim \frac{1}{3^j\ell(I)} = \ell\left(3^jI\right)^{-1} \text{ and}$$

$$\left\|h\left(3^{-j}\ \cdot\right)\varphi_{(I)}(\cdot)\right\|_{Lip\left(1,\frac{1}{2}\right)} \lesssim$$

$$\left\|h\left(3^{-j}\ \cdot\right)\right\|_{Lip\left(1,\frac{1}{2}\right)} \cdot \left\|\varphi_{(I)}\right\|_{\infty,\text{supp } h\left(3^{-j}\cdot\right)}$$

$$+\|h\|_\infty \left\|\varphi_{(I)}\right\|_{Lip\left(\left(1,\frac{1}{2}\right),\text{supp } h\left(3^{-j}\cdot\right)\right)} \lesssim \frac{1}{\ell(3^jI)}$$

$$\cdot\frac{1}{\sqrt{|I|}}\cdot 3^{-jM} + \left\|\nabla_{x,\sqrt{t}}\ \varphi_{(I)}\right\|_{\infty,\text{supp } h\left(3^{-j}\cdot\right)}$$

$$= C\frac{1}{\sqrt{|I|}}\ell(I)^{-1}3^{-j(M+1)}$$

where is not really necessary to use $\nabla_{\sqrt{t}}$—in fact one can use only $\nabla_x\varphi_{(I)}$ by an elementary maneuver. (The $Lip\left(1,\frac{1}{2}\right)$ norm does give this estimate.)

So

$$\left\|\tilde{\beta}_j\right\|_{Lip\left(1,\frac{1}{2}\right)} \lesssim \frac{1}{\sqrt{|I|}}\ell\left(3^jI\right)^{-1}3^{-Mj}.$$

Multiplying by 3^{Mj} gives the desired estimates for $\varphi_{(I),j} = 3^{Mj}\tilde{\beta}_j$. □

Good families of parabolic cubes are defined the same way that good families of Euclidean cubes are defined. In [6] the following definition is given for Euclidean cubes:

Def: A family of cubes $\mathfrak{G}$ is said to be good if: a) for all Q and Q' in $\mathfrak{G}$, either $Q \subseteq Q'$ or $Q' \subseteq Q$ or $Q \cap Q' = \phi$; b) if Q and Q' belong to $\mathfrak{G}$, $Q \subseteq Q'$ and $Q \ne Q'$, then $\ell(Q) \le .5\ell(Q')$. The definition can be applied as it is to parabolic cubes. Specifically a good family will have all the properties of the family of parabolic dyadic cubes without requiring that $\ell(I)=2^{-k}$ or that the corner point coordinates of the cube be parabolic dyadic rationals.

There is a parabolic version of Lemma 2.5 ([6]):

Lemma 2. *Let m be an odd positive integer and $\mathfrak{F}_m$ = the family of all m-fold dilates of parabolic dyadic cubes. Then $\mathfrak{F}_m = \bigcup_{i=1}^{m^{(d+1)}} \mathfrak{G}_{m,i}$ where the $\mathfrak{G}_{m,i}$ are disjoint (i.e. don't contain any of the same cubes) and each $\mathfrak{G}_{m,i}$ is a good family.*

Proof. The only difference here from [6] Lemma 2.5 is the time variable. So it suffices to take $d = 1$ and I such that $|I| = \ell(I)^2$. Any m-fold dilate of such a cube which is a parabolic dyadic cube will have the form

$$\left[\frac{m^2 j+s}{4^k},\frac{m^2(j+1)+s}{4^k}\right)=$$

$$\left[\frac{m^2(4j)+4s}{4^{k+1}},\frac{m^2(4j+1)+4s}{4^{k+1}}\right)$$

$$\cup\left[\frac{m^2(4j+1)+4s}{4^{k+1}},\frac{m^2(4j+2)+4s}{4^{k+1}}\right)$$

$$\cup\left[\frac{m^2(4j+2)+4s}{4^{k+1}},\frac{m^2(4j+3)+4s}{4^{k+1}}\right)$$

$$\cup\left[\frac{m^2(4j+3)+4s}{4^{k+1}},\frac{m^2(4j+4)+4s}{4^{k+1}}\right)$$

where $s\in\{0,1,2,\ldots,m^2-1\}$. There is a unique subdivision of I into four sub-cubes each of which is also an m^2 dilate of a parabolic dyadic cube. Moreover each m^2-dilate is exactly one fourth of a larger m^2-dilate. To show these can be chosen uniquely notice that $m^2 j + s = 4p$, $4p + 1$, $4p + 2$ or $4p + 3$ for some integer p. This means

$$m^2 j+s=\text{one of}\begin{cases}4(\mathrm{m}^2\ell+\mathrm{s}')\\4(\mathrm{m}^2\ell+\mathrm{s}')+1\\4(\mathrm{m}^2\ell+\mathrm{s}')+2\\4(\mathrm{m}^2\ell+\mathrm{s}')+3\end{cases}$$

for some s', $s'\in\{0,1,2,\ldots,m^2-1\}$ since p must $= m^2\ell+s'$ for some ℓ and s'.

This implies $s\equiv_{m^2} 4s', 4s'+1, 4s'+2$ or $4s'+3$. Since m^2 is odd, 4 is invertible in $\mathbb{Z}/m^2$ so $s'\equiv_{m_2}\frac{s}{4},\frac{s-1}{4},\frac{s-2}{4}$ or $\frac{s-3}{4}$. This gives the larger m^2-dilate uniquely. If $m^2 j+s\equiv_4 0$ then

$$\left[\frac{m^2 j+s}{4^k},\frac{(m^2)(j+1)+s}{4^k}\right)$$

is the left-most one quarter of

$$\left[\frac{m^2\ell+s'}{4^{k-1}},\frac{m^2(\ell+1)+s'}{4^{k-1}}\right),$$

if $m^2 j+s\equiv_4 1$, I is the left-middle one quarter of the larger interval, etc. □

Definition 1 *For I a parabolic cube in $\mathbb{R}^d, a_{(I)}(x',t)$ is said to be **adapted** to I if $supp\,\alpha_{(I)}\subseteq I$, $\int\alpha_{(I)}(x',t)\ dx'dt=0$, $\|\alpha_{(I)}\|_\infty\le\frac{1}{\sqrt{|I|}}$ and $\|\alpha_{(I)}\|_{Lip\left(1,\frac{1}{2}\right)}\le\frac{1}{\ell(I)\sqrt{|I|}}$.*

The following weighted square function estimate can be proved as in [6]:

Lemma 3. *Let $f(x',t)=\sum_{I\in\mathfrak{F}}\lambda_I\alpha_{(I)}(x',t)$ where $\mathfrak{F}$ is a finite sub-family of a family $\mathfrak{G}$ of "good" parabolic cubes and each $\alpha_{(I)}$ is adapted to I. For any $v\in L^1_{loc}(\mathbb{R}^d)$, $v\ge 0$, and any $\eta>1$,*

$$\int|f(x',t)|^2 v(x',t)\,dx'dt\le C(\eta,d)\sum_{I\in\mathfrak{F}}\frac{|\lambda_I|^2}{|I|}\cdot$$

$$\int_I v(x',t)\log^\eta\left(e+\frac{v(x',t)}{v_I}\right)dx'dt.$$

Lemma 3 is a consequence of the parabolic version of the following result of [5]:

Lemma 2.3 *Let $0<p<\infty$, $0<\overline{\eta}\le 1$ and let A be a positive number. Let $\mathfrak{G}\subseteq\mathfrak{D}$. Let $f=\sum_{I\in\mathfrak{F}}\lambda_I\alpha_{(I)}$ where the $\alpha_{(I)}$ are adapted to l and such that $f^*\in L^p(\mathbb{R}^d,vdx)$. Suppose that v is a weight for which $y_{\overline{\eta}}(Q,v)\le A$ for all $Q\in\mathfrak{G}$. Then there is a $C(p,d,\overline{\eta})<\infty$ such that*

$$\int|f^*|^p vdx\le C(p,d,\overline{\eta})A^{\frac{p}{2\overline{\eta}}}\int S^p_\Lambda(f)vdx.$$

Here $\mathfrak{D}$ are dyadic cubes, $f^*(x) = \sup |f_k(x)|$ where $f_k(x) = \frac{1}{|Q_k|} \int Q_k \; f(y)\,dy,\; x \in Q_k$ and $\ell(Q_k) = 2^{-k}$.

$$y_{\overline{\eta}}(Q,v) = \begin{cases} \dfrac{\int_Q v \log^{\overline{\eta}}\left(e + \frac{v}{v_Q}\right)}{\int_Q v} & if \int_Q v > 0 \\ 1 & if \int_Q v = 0 \end{cases}$$

and

$$S_\Lambda f(x) = \left(\sum_{x \in Q \in \mathfrak{S}} \frac{|\lambda_Q|^2}{|Q|} \right)^{\frac{1}{2}}.$$

Lemma 2.3 is valid if $\mathfrak{D}$ is replaced by any good family of cubes. Below we prove it for good parabolic cubes.

Also for any $\overline{\eta} \leq \eta$ we have

$$\int_I v \log^{\overline{\eta}}\left(e + \frac{v}{v_I}\right) \leq \int_I v \log^{\eta}\left(e + \frac{v}{v_I}\right)$$

so if $\int_I v \log\left(e + \frac{v}{v_I}\right) \leq A$ we have $\int_I v \log^{\overline{\eta}}\left(e + \frac{v}{v_I}\right) \leq A$ for any $\overline{\eta} \leq 1$.

Given the result of Lemma 2.3 for a good collection of parabolic cubes in place of $\mathfrak{D}$ one can prove Lemma 3 is valid by the following argument of J. M. Wilson:

Take $\mathfrak{F}_k = \{I : 2^k < y_1(I,v) \leq 2^{k+1}\}$. Assuming $v_1(I,v) \leq \infty$ for all I (otherwise the result is trivial) and taking $f_k(x',t) = \sum_{I \in \mathfrak{F}_k} \lambda_I \alpha_I(x',t)$ we have

$$\int |f(x',t)|^2 v(x',t)\,dx'dt = \int \left| \sum_k f_k(x',t) \right|^2 \cdot v(x',t)\,dx'dt$$

$$\leq C_\epsilon \sum_k 2^{k\epsilon} \int |f_k(x',t)|^2 \, v(x',t)\,dx'dt$$

$$\leq C_\epsilon \sum_k 2^{k\epsilon} \left(\sum_{I \in \mathfrak{F}_k} \frac{|\lambda_I|^2}{|I|} \right) \cdot 2^{k+1} \int_I v(x',t)\,dx'dt$$

from Lemma 2.3. Now

$$2^k < \frac{\int_I v \log\left(e + \frac{v}{v_I}\right)}{\int_I v}$$

so the above is

$$\leq C'_\epsilon \sum_k \sum_{I \subseteq \mathfrak{F}_k} \frac{|\lambda_I|^2}{|I|} \cdot \left(\frac{1}{v(I)} \int_I v \log\left(e + \frac{v}{v_I}\right) \right)^{1+\epsilon} v(I)$$

$$\leq C'_\epsilon \sum_k \sum_{I \subseteq \mathfrak{F}_k} \frac{|\lambda_I|^2}{|I|} \int_I v(x',t) \cdot \log^{1+\epsilon}\left(e + \frac{v(x',t)}{v_I}\right) = C'_\epsilon \sum_{I \in \mathfrak{F}} \frac{|\lambda_I|^2}{|I|} v(I, 1+\epsilon)$$

where the second to last inequality is from Holder's inequality. □

To see that Lemma 2.3 is valid for parabolic cubes I we first note that the condition $\int_I v \log\left(e + \frac{v}{v_I}\right) \leq A \int_I v$ for all cubes $I \Rightarrow v \in A^\infty(dx'dt)$. The proof is identical with the proof for Euclidean cubes.

Proof of the parabolic version of Lemma 2.3: One uses the exponential square theorem for parabolic dyadic martingales. The result is true for any "good" family of cubes in place of dyadic cubes. This result implies that

$$|\{f^* > 2\lambda, Sf \leq \gamma\lambda\}| \leq C_1 \, e^{-c_2\gamma^{-2}} \, |\{f^* > \lambda\}|$$

where

$$f^*(x',t) = \sup_k \; |f_k(x',t)|$$

$$f_k(x',t) = \sum_i \frac{1}{|I_k^i|} \int_{I_k^i} f(y',s)\,dy'ds \cdot \chi_{I_k}^{\,i}(x',t),$$

$$\ell(I_k^i) = 2^{-k}$$

$$d_{I_{k-1}^i} = (f_k(x',t) - f_{k-1}(x',t))\, \chi_{I_{k-1}^i}(x',t)$$

$$Sf(x',t) = \left(\sum_{\substack{k \\ (x',t) \in I_k^{i_k}}} \frac{\|d_{I_k^{i_k}}\|_2^2}{|I_k^{i_k}|} \right)^{\frac{1}{2}}$$

We have $|f(x',t)| \leq f^*(x',t)$ and $Sf(x',t) \leq c(d)\, S_\Lambda f(x',t)$. The first inequality is from the Lebesgue differentiation theorem---valid for parabolic cubes; the second is proved in [5] for dyadic Euclidean cubes. The same proof holds for good parabolic cubes.

The exponential square theorem is local, so for any cube Q_λ^i

$$\left|\left\{(x',t)\in Q_\lambda^i : f^*(x',t)>2\lambda, Sf(x',t)\le\gamma\lambda\right\}\right| \le C_1 e^{-c_2\gamma^{-2}}\left|\left\{(x',t)\in Q_\lambda^i : f^*(x',t)>\lambda\right\}\right|.$$

Taking Q_λ^i to be the maximal good cube such that $f_{Q_\lambda^i}>\lambda$ and $\sum_Q \frac{|\lambda_Q|^2}{|Q|}\le\gamma^2\lambda^2$, $Q\not\supseteq Q_\lambda^i$, $Q\in\mathfrak{G}$, and using exactly the same reasoning as in the proof of Lemma 2.3 [5] we can reduce showing

$$v(\{(x',t): f^*(x',t)>2\lambda, S_\Lambda f(x',t)\le\gamma\lambda\}) \le \epsilon(p)\ v(\{(x',t): f^*(x',t)>\lambda\})$$

to showing

$$v\left(\left\{(x',t)\in Q_k : \left(f-f_{Q_\lambda^i}\right)^* > .8\lambda, S_\Lambda f(x',t)\le\gamma\lambda\right\}\right) \le \epsilon(p)\ v(Q_k)$$

where Q_k are the maximal subcubes of Q_λ^i that are also in $\mathfrak{G}$.

But this last inequality follows from the fact that $v\in A^\infty(dx'dt)$, $Sf(x',t)\le C(d)\ S_\Lambda f(x',t)$ and the exponential square good λ inequality on the cubes Q_k for Lebesgue measure (Q_k are disjoint).

The fact that

$$\sum_{\substack{I\\ Q_\lambda^i\subsetneq I\in\mathfrak{G}}}\frac{|\lambda_I|^2}{|I|}\le C(d)\gamma^2\lambda^2$$

implies that

$$\left|f_{Q_\lambda^i}\right|\le\left|f_{\tilde{Q}}-f_{Q_\lambda^i}\right|+\left|f_{\tilde{Q}}\right|\lesssim\sqrt{C(d)}\ \gamma\lambda+\lambda$$

where $\tilde{Q}\supsetneq Q_\lambda^i$, $\ell(\tilde{Q})=2\ell(Q_\lambda^i)$, by the following estimate:

$$\left|f_{\tilde{Q}}-f_{Q_\lambda^i}\right|=\left|⨍_{\tilde{Q}} f(x',t)\,dx'dt-⨍_{Q_\lambda^i}\sum\lambda_I\alpha_{(I)}\right|\le$$

$$\left|⨍_{Q_\lambda^i}\sum_{\substack{I\\ Q_\lambda^i\subsetneq I\in\mathfrak{G}}}\lambda_I\alpha_{(I)}(y',s)-⨍_{\tilde{Q}}\sum_{\substack{I\\ Q_\lambda^i\subsetneq I\in\mathfrak{G}}}\lambda_I\alpha_{(I)}(x',t)\right|$$

$$\le⨍_{\tilde{Q}}⨍_{Q_\lambda^i}\sum_{Q_\lambda^i\subsetneq I\in\mathfrak{G}}|\lambda_I|\left|\alpha_{(I)}(y',s)-\alpha_{(I)}(x',t)\right|dy'ds\,dx'dt$$

$$\le\sum_{Q_\lambda^i\subsetneq I\in\mathfrak{G},\,x\in\tilde{Q}}|\lambda_I|\left\|\alpha_{(I)}\right\|_{Lip\left(1,\frac{1}{2}\right)}\ell(\tilde{Q})$$

$$\le\sum_{Q_\lambda^i\subsetneq I}\frac{|\lambda_I|}{\sqrt{|I|}}\frac{\ell(Q)}{\ell(I)}$$

$$\le\left(\sum\frac{|\lambda_I|^2}{|I|}\right)^{\frac{1}{2}}\left(\sum_{I\supseteq\tilde{Q}}\left(\frac{\ell(\tilde{Q})}{\ell(I)}\right)^2\right)^{\frac{1}{2}}$$

$$\le C\left(\sum_{x\subset\tilde{Q}}\frac{|\lambda_I|^2}{|I|}\right)^{\frac{1}{2}}\le C\sqrt{C(d)}\gamma\lambda$$

since $\sum\left(\frac{\ell(\tilde{Q})}{\ell(I)}\right)^2$ is a geometric series.

Q_λ^i is maximal implies that $\left|f_{\tilde{Q}}\right|\le\lambda$ so we get $\left|f_{Q_\lambda^i}\right|\le 1.1\lambda$ if $C\cdot C(d)\gamma\le .1$

Now one can show that

$$\begin{aligned}&\left\{(x',t)\in Q_\lambda^i : f^*(x',t)>2\lambda, S_\Lambda f(x',t)\le\gamma\lambda\right\}\\ &\subseteq\bigcup_k\left\{Q_k : \left(f-f_{Q_k}\right)^*>.8\lambda, S_\Lambda f(x',t)\le\gamma\lambda\right\}\end{aligned}\quad(**)$$

by noting that if $f^*(x',t)>2\lambda$, there is a cube Q such that $f_Q>2\lambda$ and $(x',t)\in Q$. Since Q_λ^i is maximal, Q must $\subseteq Q_\lambda^i$. Now one can show for any $Q\subseteq Q_\lambda^i$, such that $Q\not\subset Q_k$ for any $Q_k\in\{Q\in\mathfrak{G}$ such that Q is maximal in $Q_\lambda^i\}=\partial\ell$, that $\left|f_Q-f_{Q_\lambda^i}\right|\le C\cdot C(d)\gamma\lambda$ by almost the same proof given above for $Q=\tilde{Q}$ (Q can be any cube, including one of the cubes in $\partial\ell$, such that $Q\not\subset Q_k$). Then no such Q can be one where $\left|f_Q\right|>2\lambda$ since $\left|f_Q\right|\le .1\lambda+1.1\lambda=1.2\lambda$. So any (x',t) such that $f^*(x',t)>2\lambda$ must lie inside a Q_k. Also

$$f^*(x',t)>2\lambda\Leftrightarrow\left(f^*-f_{Q_k}\right)+f_{Q_k}-f_{Q_\lambda^i}+f_{Q_\lambda^i}>2\lambda$$

if $(x',t)\in Q_k$ so

$$\left(f-f_{Q_k}\right)^*>2\lambda-\left|f_{Q_k}-f_{Q_\lambda^i}\right|-\left|f_{Q_\lambda^i}\right|\ge 2\lambda-.1\lambda-1.1\lambda=.8\lambda.$$

This is enough to guarantee (∗ ∗) holds. □

Given Lemma 3 one can now prove Theorem 1 as in [6]:

Proof: Let $\mathfrak{F}_j$ be the collection of all 3^j-fold dilates of parabolic dyadic cubes. By Lemma 2 above $\mathfrak{F}_j = \bigcup_{i=1}^{3^{(d+1)j}} \mathfrak{G}_{j,i}$ where each $\mathfrak{G}_{j,i}$ is a good family and no two $\mathfrak{G}_{j,i}$ contain the same cube. Taking $f(x',t) = \sum_{I\in\mathfrak{F}} \lambda_I \varphi_{(I)}(x',t)$ and applying Lemma 1 to each $\varphi_{(I)}$ gives

$$f(x',t) = C\sum_{I\in\mathfrak{F}}\lambda_I \sum_{j=0}^{\infty} 3^{-Mj}\varphi_{(I),j}(x',t)$$

where supp $\varphi_{(I),j} \subseteq 3^j I, \int \varphi_{(I),j} = 0$, etc.
So

$$f(x',t) = C(M,d)\sum_{j=0}^{\infty}\sum_{i=1}^{3^{j(d+1)}}\sum_{3^j I\in\mathfrak{G}_{j,i}, I\in\mathfrak{F}} \lambda_I 3^{-Mj} 3^{\frac{(d+1)j}{2}} \widetilde{\varphi}_{(I),j}(x',t)$$

where the $\widetilde{\varphi}_{(I),j}$ are easily seen to be adapted to $3^j I$. Now

$$\int_{\mathbb{R}^d} |f(x',t)|^2 v(x',t)\,dx'dt$$
$$= C\int_{\mathbb{R}^d}\left|\sum_{j=0}^{\infty}\sum_{i=1}^{3^{j(d+1)}}\sum_{3^j I\in\mathfrak{G}_{j,i}, I\in\mathfrak{F}} \lambda_I 3^{-Mj} 3^{\frac{(d+1)j}{2}} \widetilde{\varphi}_{(I),j}(x',t)\right|^2 \cdot$$
$$v(x',t)\,dx'dt \le C(M,d)\int_{\mathbb{R}^d}\left(\sum_{j=0}^{\infty} 3^{-j\epsilon}\right)^{\frac{1}{2}\cdot 2}\cdot$$
$$\left(\sum_{j=0}^{\infty} 3^{+j\frac{\epsilon}{2}\cdot 2}\left|\sum_{i=1}^{3^{j(d+1)}}\sum_{\mathfrak{G}_{j,i,\mathfrak{F}}} \lambda_I 3^{-Mj} 3^{j\frac{(d+1)j}{2}} \widetilde{\varphi}_{(I),j}(x',t)\right|^2\right)^{\frac{1}{2}\cdot 2}\cdot$$
$$v(x',t)\,dx'dt \le C(M,d,\epsilon)\sum_{j=0}^{\infty} 3^{j(d+1+\epsilon)}\sum_{i=1}^{3^{j(d+1)}}\cdot$$
$$\int_{\mathbb{R}^d}\left|\sum_{3^j I\in\mathfrak{G}_{j,i}, I\in\mathfrak{F}} \lambda_I 3^{-Mj} 3^{\frac{(d+1)j}{2}} \widetilde{\varphi}_{(I),j}(x',t)\right|^2\cdot$$
$$v(x',t)\,dx'dt$$

Using Lemma 3 this last quantity is

$$\le C(M,d,\epsilon)\sum_{j=0}^{\infty} 3^{j(d+1+\epsilon)}\sum_{i=1}^{3^{j(d+1)}}\sum_{3^j I\in\mathfrak{G}_{j,i}, I\in\mathfrak{F}} \frac{|\lambda_I|^2}{|I|}\cdot$$
$$3^{-2Mj} v(3^j I,\eta)$$
$$= C(M,d,\epsilon)\sum_{I\in\mathfrak{F}}\frac{|\lambda_I|^2}{|I|}\sum_{j=0}^{\infty} 3^{-j(2M-d-1-\epsilon)} v(3^j I,\eta) \;\square$$

The Littlewood-Paley inequality of Theorem 1 can now be used to prove the following conditions on measures $d\mu(x',t)$ and $v(x',t)\cdot\ dx'dt$ to obtain the weighted inequalities for the heat equation on the right half plane and on the box domain.

Theorem 2. *Suppose* $\nabla u(x,t) = \int_{\partial_p\Omega} K(x,t;y',s)\, f(y',s)\, dy'ds$ *where the kernel K satisfies*

$$\int_{\partial_p\Omega} K(\cdot;y',s)\,dy'ds = 0,$$

for $(x,t)\in T(I)$, $(x'_{(I)},t_{(I)})$ *is the center of I,*

$$|K(x,t;y's)| \lesssim \ell(I)^{-H}\left(1+\frac{d_p((y',s);I)}{\ell(I)}\right)^{-M},$$
$$|\nabla_{y'}K(x,t;y',s)| \le \ell(I)^{-H-1}\cdot$$
$$\left(1+\frac{d_p((y',s);(x'_{(I)},t_{(I)}))}{\ell(I)}\right)^{-M-1},$$
$$\left|\frac{\partial K}{\partial s}(x,t;y',s)\right| \le \ell(I)^{-H-2}\cdot$$
$$\left(1+\frac{d_p((y',s);(x'_{(I)},t_{(I)}))}{\ell(I)}\right)^{-M-2}$$

where $M > d+1$.

For $\mu(x,t)$ *a Borel measure,* $v(x',t)\ge 0$, $v\in L^1_{loc}(\mathbb{R}^d)$ *and for v satisfying (W) below, assume for all dyadic parabolic cubes* $I\subseteq R^d$ *we have*

$$\mu(T(I))^{\frac{1}{q}}\left(\int_{\mathbb{R}^d}\frac{w(x',t)\,dx'dt}{\left(1+\frac{d_p(x',t;x'_{(I)},t_{(I)})}{\ell(I)}\right)^{p'M-\frac{p'(d+1+\epsilon)}{2}}}\right)^{\frac{1}{p'}}\cdot$$
$$\le \ell(I)^H \qquad (*).$$

Then an appropriate constant $C = C(M,d,\epsilon,p,q,\tau)$ *exists so that*

$$\left(\int_{\Omega}|\nabla u(x,t)|^q d\mu(x,t)\right)^{\frac{1}{q}} \le$$
$$C\left(\int_{\partial_p\Omega}|f(x',t)|^p v(x',t)\,dx'dt\right)^{\frac{1}{p'}}$$

if $1 < p \le 2 \le q < \infty$.

(W) For some $\tau > \frac{p'}{2}$, $\frac{1}{p} + \frac{1}{p'} = 1$, *there is a weight* $w(x',t)$ *so that for all dyadic parabolic cubes I,*

$$\int_I [v(x',t)]^{1-p'} \log^\tau\left(e + \frac{(v(x',t))^{1-p'}}{\left(v^{1-p'}\right)_I} \right) dx'dt$$

$$\le \int_I w(x',t)\, dx'dt .$$

So if $u(x,t)$ is a solution of $\left(\frac{\partial}{\partial t} - \Delta\right) u = 0$ in the right half plane or in the (d + 1)-dimensional box domain with boundary data f, then the condition (∗) on μ and v gives the weighted inequality. It should be noted that the parabolic version of Wilson's Corollary 2.8 [6] is valid for a convolution kernel of the form

$$\Psi_{x_d}(x',t) = x_d^{-d-1}\Psi\left(\frac{x'}{x_d}, \frac{t}{x_d^2}\right).$$

In fact the heat kernel can be written in this form with the dilation factor appearing in front as x_d^{-M} instead of x_d^{-d-1}.

Proof of Theorem 2: Using the dual inequality, one needs to find conditions so that

$$\left(\int_{\partial\Omega} |Tg(x',t)|^{p'} \sigma(x',t)\, dx'dt \right)^{\frac{1}{p'}}$$

$$\le \left(\int_\Omega |g(x,t)|^{q'} d\mu(x,t) \right)^{\frac{1}{q'}}$$

where $\sigma(x',t) = v(x',t)^{1-p'}$

$$\int_\Omega \nabla u(x,t)\, g(x,t)\, d\mu(x,t) = \int_{\partial\Omega} f(y',s) \cdot \left(\int_\Omega K(x,t;y',s)\, g(x,t)\, d\mu(x,t) \right) dy'ds =$$

$$\int_{\partial\Omega} f(y',s)\; Tg(y',s)\, dy'ds \text{ where}$$

$$Tg(y',s) = \int_\Omega K(x,t;y',s)\, g(x,t)\, d\mu(x,t).$$

So applying Holder's inequality to the last integral, one needs to bound the second term in

$$\left(\int_{\partial\Omega} |f(y',s)|^p\, v(y',s)\, dy'ds \right)^{\frac{1}{p}} .$$

$$\left(\int_{\partial\Omega} |Tg(y',s)|^{p'} \left(\frac{1}{v(y',s)} \right)^{\frac{p'}{p}} dy'ds \right)^{\frac{1}{p'}}$$

$$\left(\frac{1}{p} + \frac{1}{p'} = 1 \Rightarrow \frac{p'}{p} = p' - 1 \right)$$

Now

$$Tg(y',s) = \sum_{I \in \mathfrak{D}} \int_{T(I)} K(x,t;y',s)\, g(x,t)\, d\mu(x,t)$$

$$= \sum_{I \in \mathfrak{D}} \lambda_I \varphi_{(I)}(y',s) \text{ where}$$

$$\lambda_I \le \ell(I)^{\frac{d+1}{2} - H} \left(\int_{T(I)} g^{q'} d\mu \right)^{\frac{1}{q'}} \cdot \mu(T(I))^{\frac{1}{q}}.$$

Then for K as in the theorem statement, $\varphi_{(I)}(y',s)$ satisfies the conditions given in (1) - (3) (see the estimates on K below). Using the result of Theorem 1 on $\sum \varphi_{(I)}\lambda_I$ (on supp g, which can be assumed to be compact in Ω, so only finitely many terms are non-zero) gives

$$\int_{\partial\Omega} |Tg|^{p'} \sigma dx'dt$$

$$\le C \sum_{I \in \mathfrak{G}} \frac{|\lambda_I|^2}{|I|} \sum_{j=0}^{\infty} 3^{-j(2M-d-1-\epsilon)}\, \sigma(3^j I; \eta)$$

(if p = 2).

To estimate the L^p norm we want to find

$$\sup_{\|h\|_{L^{(p'/2)'}(\sigma dx'dt)} = 1} \int |Tg|^2 h\sigma dx'dt = \left(\int |Tg|^{2 \cdot \frac{p'}{2}} \sigma dx'dt \right)^{\frac{2}{p'}}$$

By Theorem 1

$$\int |Tg|^2 h\sigma dx'dt$$

$$\le C \sum \frac{|\lambda_I|^2}{|I|} \sum 3^{-j(2M-d-1-\epsilon)}\, h\sigma(3^j I; \eta), (\eta > 1)$$

and using estimates on Orlicz norms $h\sigma(3^j I; \eta) \le C\sigma\left(3^j I; \eta \cdot \frac{p'}{2}\right)^{\frac{2}{p'}}$ — the proof of this is due to J. M. Wilson and the identical proof is valid if I is parabolic instead of Euclidean, since it involves Orlicz norms, not geometry.

For τ such that $\tau = \eta \frac{p'}{2}$ (η must be > 1 so $\frac{2\tau}{p'} > 1$ or $\tau > \frac{p'}{2}$) by hypothesis

$$\sigma(3^j I; \tau)^{\frac{2}{p'}} \le w(3^j I)^{\frac{2}{p'}} \text{ so}$$

$$\int |Tg|^2 h\sigma dx'dt$$

$$\le C\sum \frac{|\lambda_I|^2}{|I|}\sum_{j=0}^{\infty} 3^{-j(2M-d-1-\epsilon)} w\left(3^j I\right)^{\frac{2}{p'}}$$

and replacing ϵ by $\epsilon/4$ gives

$$\sum_{j=0}^{\infty} 3^{-j\left(2M-d-1-\frac{\epsilon}{4}\right)} w\left(3^j I\right)^{\frac{2}{p'}}$$

$$\le C_{\epsilon,p'}\left(\sum_{j=0}^{\infty} 3^{-j(2M-d-1-\epsilon)\cdot\frac{p'}{2}} w\left(3^j I\right)\right)^{\frac{2}{p'}}$$

$$\le C_{\epsilon,p'}\left(\int_{\mathbb{R}^d} \frac{w(x',t)\,dx'dt}{\left(1+\frac{d_p\left((x',t);\left(x'_{(I)},t_{(I)}\right)\right)}{\ell(I)}\right)^{p'M-\frac{p'}{2}(d+1+\epsilon)}}\right)^{\frac{2}{p'}}$$

by dividing $\mathbb{R}^d$ into parabolic annuli of dimension $3^j\ell(I), j = 0,1,2,3,\ldots,\infty$ and writing the sum $\sum_{j=0}^{\infty} 3^{-j(2M-d-1-\epsilon)\frac{p'}{2}} w\left(3^j I\right)$ as

$$\sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty} 3^{-k(2M-d-1-\epsilon)\frac{p'}{2}}\right)\left(w\left(3^j I\setminus 3^{j-1}I\right)\right)$$

and making the obvious estimate on $3^j I\setminus 3^{j-1}I$ ($3^{j-1}I=\phi$ here).

So $\int_{\mathbb{R}^d}|Tg|^2 h\sigma dx'dt \lesssim$

$$\sum_I \frac{|\lambda_I|^2}{|I|}\left(\int_{\mathbb{R}^d} \frac{w(x',t)\,dx'dt}{\left(1+\frac{d_p\left((x',t);\left(x'_{(I)},t_{(I)}\right)\right)}{\ell(I)}\right)^{p'M-\frac{p'}{2}(d+1+\epsilon)}}\right)^{\frac{2}{p'}}$$

$$=\sum_I \frac{|\lambda_I|^2}{|I|}\left(w\left(x'_{(I)},t_{(I)}\right)\right)^{\frac{2}{p'}}$$

$$\le\left(\sum_I\left(\int_{T(I)} g^{q'}d\mu\right)^{\frac{2}{q'}}(\mu(T(I)))^{\frac{2}{q}}\cdot \frac{\ell(I)^{-2H+\frac{d+1}{2}\cdot 2}}{|I|} w\left(x'_{(I)},t_{(I)}\right)^{\frac{2}{p'}}\right)$$

$$\le\left(\sum_I\left(\int_{T(I)} g^{q'}d\mu\right)^{\frac{2}{q'}}\right)\le\left(\sum_I\int_{T(I)} g^{q'}d\mu\right)^{\frac{2}{q'}} \text{ if}$$

$$\mu(T(I))^{\frac{1}{q}}\left(\int_{\mathbb{R}^d} \frac{w(x',t)\,dx'dt}{\left(1+\frac{d_p\left(x',t;x'_{(I)},t_{(I)}\right)}{\ell(I)}\right)^{p'M-\frac{p'}{2}(d+1+\epsilon)}}\right)^{\frac{1}{p'}}\le \ell(I)^H \blacksquare$$

The same result is valid for $p\ge 2$ also $(p\le q<\infty)$ by a more general version of Theorem 1. The proof is almost identical with the proof of Theorem 2.10 in [6] and will be omitted.

Estimates on the heat kernel:

Solutions to the heat equation with boundary function $f(x',t)$ will be of the form

$$u(x,t)=\int_{\mathbb{R}^d} K(\vec{x},t;y',0,s)\,f(y',s)\,dy'ds$$

where

$$K(\vec{x},t;\vec{y},s)=\frac{\partial W}{\partial y_d}(\vec{x},t;\vec{y},s)\,\Big|_{y_d=0}$$

$$=\frac{x_d}{2(4\pi)^{\frac{d}{2}}(t-s)^{\frac{d+2}{2}}}\exp\left\{-\frac{|\vec{x}-y'|^2}{4(t-s)}\right\} \text{ if } t>s$$

on the right half space, $\mathbb{R}_r^{d+1}$, where $(x',x_d)=\vec{x}=(x_1,x_2,\ldots,x_{d-1},x_d), \partial\mathbb{R}_r^{d+1}=\mathbb{R}^d=\left\{(x',t)\,\middle|\,x'\in\mathbb{R}^{d-1}, t\in\mathbb{R}^1\right\}$ so $x_d>0$ in $\mathbb{R}^{d+1}$,

$$W(\vec{x},t;\vec{y},s)$$

$$=\begin{cases}\frac{1}{(4\pi)^{\frac{d}{2}}(t-s)^{\frac{d}{2}}}\exp\left\{\frac{-|\vec{x}-\vec{y}|^2}{4(t-s)}\right\} & \text{if } t>s\\ 0 & \text{if } t\le s\end{cases}$$

To obtain the estimates on ∇u, the kernel to which we are applying Theorem 2 will be any/all of the following:

$$\underset{i\ne d}{\frac{\partial K}{\partial x_i}}(\vec{x},t;y',0,s)$$

$$=\begin{cases}\frac{-(x_i-y_i)x_d}{4(4\pi)^{\frac{d}{2}}(t-s)^{\frac{d+4}{2}}}\exp\left\{\frac{-|\vec{x}-\vec{y}|^2}{4(t-s)}\right\} & \text{if } t>s\\ 0 & \text{if } t\le s\end{cases}$$

$$\frac{\partial K}{\partial x_d}(\vec{x},t;y',0,s)$$

$$=\begin{cases}\frac{2(t-s)-x_d^2}{4(4\pi)^{\frac{d}{2}}(t-s)^{\frac{d+4}{2}}}\exp\left\{\frac{-|\vec{x}-y'|^2}{4(t-s)}\right\} & \text{if } t>s\\ 0 & \text{if } t\le s\end{cases} \quad \text{and}$$

$$\frac{\partial K}{\partial t}(\vec{x},t;y',0,s)$$

$$=\begin{cases}\frac{x_d|x'-y'|^2+x_d^3-2(d+2)(t-s)x_d}{8(4\pi)^{\frac{d}{2}}(t-s)^{\frac{d+6}{2}}}\exp\left\{\frac{-|\vec{x}-y'|^2}{4(t-s)}\right\} & \text{if } t>s\\ 0 & \text{if } t\le s\end{cases}$$

To show these kernels satisfy the estimates needed for Theorem 2 we need to estimate the second derivatives:

$$\frac{\partial^2 K}{\partial x_i\,\partial y_j}(\vec{x},t;y',0,s)\quad j\neq i$$

$$=\begin{cases}\dfrac{-(x_i-y_i)(x_j-y_j)x_d}{8(4\pi)^{\frac{d}{2}}(t-s)^{\frac{d+6}{2}}}\exp\left\{\dfrac{-|\vec{x}-y'|^2}{4(t-s)}\right\} & if\ t>s\\ 0 & if\ t\le s\end{cases}$$

$$\frac{\partial^2 K}{\partial x_i\partial y_i}(\vec{x},t;y',0,s)\quad for\ i<d$$

$$=\begin{cases}\dfrac{\left(2(t-s)-(x_i-y_i)^2\right)x_d}{8(4\pi)^{\frac{d}{2}}(t-s)^{\frac{d+6}{2}}}\exp\left\{\dfrac{-|\vec{x}-y'|^2}{4(t-s)}\right\} & if\ t>s\\ 0 & if\ t\le s\end{cases}$$

$$\frac{\partial^2 K}{\partial x_d\,\partial y_j}(\vec{x},t;y',0,s)\quad j<d$$

$$=\begin{cases}\dfrac{\left(2(t-s)-x_d^2\right)(x_j-y_j)}{8(4\pi)^{\frac{d}{2}}(t-s)^{\frac{d+6}{2}}}\exp\left\{\dfrac{-|\vec{x}-y'|^2}{4(t-s)}\right\} & if\ t>s\\ 0 & if\ t\le s\end{cases}$$

$$\frac{\partial^2 K}{\partial t\partial y_j}(\vec{x},t;y',0,s)$$

$$=\begin{cases}\left[\dfrac{(x_j-y_j)\left(|\vec{x}-y'|^2x_d-2(d+4)(t-s)x_d\right)}{16(4\pi)^{\frac{d}{2}}(t-s)^{\frac{d+8}{2}}}\right]\exp\left\{\dfrac{-|\vec{x}-y'|^2}{4(t-s)}\right\} & if\ t>s\\ 0 & if\ t\le s\end{cases}$$

$$\frac{\partial^2 K}{\partial t\partial s}(\vec{x},t;y',0,s)$$

$$=\left\{\frac{\left[4(d+4)|\vec{x}-y'|^2(t-s)-4(d+6)(d+2)(t-s)^2-|\vec{x}-y'|^4\right]x_d}{32(4\pi)^{\frac{d}{2}}(t-s)^{\frac{d+10}{2}}}\right.\cdot$$

$$\exp\left\{\frac{-|\vec{x}-y'|^2}{4(t-s)}\right\}\ if\ t>s$$

$$0\ \ if\ t\le s$$

$$\frac{\partial^2 K}{\partial x_i\partial s}(\vec{x},t;y',0,s)$$

$$\left[\frac{(x_i-y_i)x_d\left(|\vec{x}-y'|^2-2(d+4)(t-s)\right)}{16(4\pi)^{\frac{d}{2}}(t-s)^{\frac{d+8}{2}}}\right]$$

$$\cdot\exp\left\{\frac{-|\vec{x}-y'|^2}{4(t-s)}\right\}\ for\ t>s,\ 0\ if\ t\le s$$

$$\frac{\partial^2 K}{\partial x_d\partial s}=$$

$$\frac{\left[4(d+2)(t-s)^2+|\vec{x}-y'|^2x_d^2-2(d+4)x_d^2(t-s)-2|x-y'|^2(t-s)\right]}{16(4\pi)^{\frac{d}{2}}(t-s)^{\frac{d+8}{2}}}\cdot$$

$$\exp\left\{\frac{-|\vec{x}-y'|^2}{4(t-s)}\right\}\ for\ t>s, 0\ if\ t\le s.$$

Writing all of the above in the form

$$c_1(t-s)^{\frac{-\ell}{2}}\cdot\left[\frac{1}{\left(\frac{(t-s)^{\frac{1}{2}}}{x_d}\right)^{M_1}}\cdot\frac{1}{\left(\frac{(t-s)^{\frac{1}{2}}}{(x_j-y_j)}\right)^{M_2}}+c_2\right]\cdot$$

$$\exp\left\{\frac{-|\vec{x}-y'|^2}{4(t-s)}\right\}$$

the idea is that the exponential factor will control any other factor that starts to blow up. When $(x',x_d,t)\in T(I)$ then $x_d\sim\ell(I)$ so for $(t-s)^{\frac{1}{2}}>\max\left(|x_j-y_j|,x_d\right)$, we have a decay estimate of

$$C\left(\frac{(t-s)}{x_d^2}\right)^{\frac{-\ell}{2}}\cdot\frac{1}{x_d^\ell}\sim\left(\frac{(t-s)^{\frac{1}{2}}}{\ell(I)}\right)^{-\ell}\cdot\frac{1}{\ell(I)^\ell}.$$

If $(t-s)$ and $\max|x_j-y_j|$ are both small compared to x_d, the factor of $\left(\frac{(t-s)^{\frac{1}{2}}}{\ell(I)}\right)^{-\ell}$ can be put with the exponential term to get a bound of $C_{\ell,M_i}\ell(I)^{-\ell}$.

For $(t-s)^{\frac{1}{2}}<\max\limits_j|x_j-y_j|$ we can write

$$(t-s)^{\frac{-\ell}{2}}=\left(\frac{(t-s)^{\frac{1}{2}}}{(x_j-y_j)}\right)^{-\ell}\cdot\frac{1}{\left(\frac{|x_j-y_j|}{x_d}\right)^\ell}\cdot\frac{1}{x_d^\ell}.$$

Since the exponential $e^{\frac{-|\vec{x}-y'|^2}{4(t-s)}}$ damps off any factor of $\left(\frac{|x_j-y_j|^2}{(t-s)}\right)^m$ with a constant depending on m, for $t-s$ small compared with the spatial dimension on $\partial\Omega(\mathbb{R}^d)$ the kernel is

$$\le C_{m,d}\frac{1}{\left(\frac{|x_j-y_j|}{\ell(I)}\right)^\ell}\cdot\frac{1}{\ell(I)^\ell}.$$

Notice in the notation of Theorem 2 $M = H = \ell$ here.

Altogether it is easy to see that the above kernels satisfy for any $(y',0,s) \in \partial\Omega$, $(\vec{x},t) \in T(I)$

$$\left|\hat{K}(\vec{x},t,y',0,s)\right| \leq$$

$$C(H,d)\left(1+\frac{d_p\left((y',0,s);\left(x'_{(I)},t_{(I)}\right)\right)}{\ell(I)}\right)^{-H}\cdot\frac{1}{\ell(I)^H}$$

and their derivatives satisfy

$$\left|\frac{\partial\hat{K}}{\partial y_j}(\vec{x},t,y',0,s)\right| \leq$$

$$C(H,d)\frac{1}{\ell(I)^{H+1}}\left(1+\frac{d_p\left((y',0,s);\left(x'_{(I)},t_{(I)}\right)\right)}{\ell(I)}\right)^{-H-1}$$

$$\left|\frac{\partial\hat{K}}{\partial s}(\vec{x},t;y',0,s)\right| \leq$$

$$C(H,d)\frac{1}{\ell(I)^{H+2}}\left(1+\frac{d_p\left((y',0,s);\left(x'_{(I)},t_{(I)}\right)\right)}{\ell(I)}\right)^{-H-2}$$

For $\hat{K} = \frac{\partial K}{\partial x_i}, i = 1,2,\ldots,d, H = d+2$. For $\hat{K} = \frac{\partial K}{\partial t}$ the decay order is $H = d+3$.

Since the parabolic dimension of $\mathbb{R}^d = \partial\Omega$ is d + 1 there is ample decay to apply Theorem 2.

$\frac{\partial K}{\partial x_i}$ is odd in the y_i variable if $i < d$ so $\int \frac{\partial K}{\partial x_i} = 0$.

$\frac{\partial K}{\partial t} = -\frac{\partial K}{\partial s}$ and the decay in K implies $\int \frac{\partial K}{\partial s} = 0$.

Finally

$$\frac{\partial K}{\partial x_d} = -\frac{\partial^2 W}{\partial y_d^2} = -\frac{\partial W}{\partial s} + \sum_{i<d}\frac{\partial^2 W}{\partial y_i^2}$$

so $\int \frac{\partial K}{\partial x_d} = 0$ as well. So $\hat{K}(x,t;y',s)$ satisfies all conditions needed to apply Theorem 2.

Theorem 2 for the box domain R can be proved using the collection of dyadic parabolic cubes in each side $S_i = \{(\hat{x}^i,t): \hat{x}^i = x' \in \mathbb{R}^{d-1}$ with $x^i = 0$ or 1_j $0 < x_k < 1$ for $k \neq i, -\infty < t < \infty\}$. The cubes have side length ≤ 1. $T(I) =$ top half of the Carleson box in R closest to I, and if $d > 1$ the "box" $T(I)$, for I close to the edge joining S_i with S_{i+1}, will be truncated so that only if $d_p(y,s;S_i) \leq d_p(y,s;S_{i+1})$ will $(y,s) \in T(I)$ $(I \subseteq S_i \text{ here})$. The measures μ and $v(\hat{x}^i,t)\,d\hat{x}^i dt$ are defined in R and on S_i respectively. In fact one can use the versions of Lemmas 1, 2 and 3 on $\mathbb{R}^d$ as they are, confining the conditions and results to the parabolic cubes in each S_i.

Also the $\varphi_{(I)}$'s will be supported on S_i. This does not create a problem with the support of $\varphi_{(i),j}$ in Lemma 1 being on $\mathbb{R}^d$. The condition for μ and v will be

$$\mu(T(I))^{\frac{1}{q}}\left(\int_{S_i^d}\frac{w(\hat{x}_i,t)\,d\hat{x}^i dt}{\left(1+\frac{d_p\left(\hat{x}_i,t;x'_{(I)},t_{(I)}\right)}{\ell(I)}\right)^{p'M-\frac{p'}{2}(d+1+\epsilon)}}\right)^{\frac{1}{p'}}$$

$$\leq C\ell(I)^H$$

for $I \in \mathfrak{PD} = \bigcup_{i=1}^d$ {parabolic dyadic subcubes of S_i of length ≤ 1, $i = 1,2,\ldots,d$. }

The conclusion of Theorem 2 is of course that

$$\left(\int_R |\nabla u(x,t)|^q\, d\mu(x,t)\right)^{\frac{1}{q}} \leq$$

$$C\left(\int_{\partial_p R}|f(x',t)|^p\, v(x',t)\,dx'dt\right)$$

for an appropriate constant $C = C(M,d,\epsilon,p,q,\tau)$ and (W) is only needed on $I \in \mathfrak{PD}$.

All that is left is to show that the kernel K on the box domain R satisfies the right conditions for applying Theorem 2. Using Hattemer's formulas [2] for the heat kernel on R, in the case of $R \subseteq \mathbb{R}^2$,

$$K(x,t;y_i,s) = G_2(x,t;y_i,s)$$

$$= \begin{cases} \pm\frac{\partial G_1}{\partial y}(x,t;y_i,s) & \text{if } t > s \\ 0 & \text{if } t \leq s \end{cases} \quad \text{for } y_i = 0,1.$$

The sign will be chosen in this paper to correspond to the inward normal derivative. For $\theta(x,t) = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}}\left[\exp\left\{-\frac{(x+2k)^2}{4t}\right\}\right]$ we have

$$G_1(x,t;y,s) = \frac{1}{2}\left[\,\theta(x-y,t-s) - \theta(x+y,t-s)\,\right]$$

$$=\frac{1}{2}\sum_{k=-\infty}^{\infty}\frac{1}{\sqrt{4\pi(t-s)}}\cdot$$

$$\left[\exp\left\{\frac{-(x-y+2k)^2}{4(t-s)}\right\}-\exp\left\{\frac{-(x+y+2k)^2}{4(t-s)}\right\}\right]\text{f}$$

or $t > s$, so

$$\frac{\partial G_1}{\partial y}(x,t;y,s)=\frac{1}{2}\sum_{k=-\infty}^{\infty}\frac{1}{2\sqrt{\pi(t-s)}}\cdot$$

$$\left[\frac{2(x-y+2k)}{4(t-s)}\cdot\exp\left\{\frac{-(x-y+2k)^2}{4(t-s)}\right\}\right.$$

$$\left.+\frac{2(x+y+2k)}{4(t-s)}\exp\left\{\frac{-(x+y+2k)^2}{4(t-s)}\right\}\right]\text{ for } t>s.$$

$$G_2(x,t;0,s)=\frac{1}{\sqrt{4\pi(t-s)}}\cdot$$

$$\sum_{k=-\infty}^{\infty}\frac{(x+2k)}{\sqrt{(t-s)}}\exp\left\{\frac{-(x+2k)^2}{4(t-s)}\right\}\textit{for } t>s$$

$$G_2(x,t;1,s)=\frac{-1}{\sqrt{4\pi(t-s)}}\cdot$$

$$\sum_{k=-\infty}^{\infty}\left[\frac{1}{2}\frac{(x+2k-1)}{(t-s)^{\frac{1}{2}}}\exp\left\{\frac{-(x+2k-1)^2}{4(t-s)}\right\}\right.$$

$$\left.+\frac{(x+2k+1)}{2(t-s)^{\frac{1}{2}}}\exp\left\{\frac{-(x+2k+1)^2}{4(t-s)}\right\}\right]$$

$$=\frac{1}{4\sqrt{\pi}(t-s)}\sum_{k=-\infty}^{\infty}\frac{(1-x+2k)}{\sqrt{(t-s)}}\cdot$$

$$\cdot\exp\left\{\frac{-(1-x+2k)^2}{4(t-s)}\right\}\textit{for } t>s, 0 \textit{ for } t\le s$$

At $y=0,\ K(x,t;0,s)=$

$$\begin{cases}\dfrac{1}{4\sqrt{\pi}(t-s)}\cdot\displaystyle\sum_{k=-\infty}^{\infty}\frac{x+2k}{\sqrt{(t-s)}}\exp\left\{\frac{-(x+2k)^2}{4(t-s)}\right\} & \textit{for } t>s\\ 0 \textit{ for } t\le s\end{cases}$$

and at $y=1$ $K(x,t;1,s)=$

$$\begin{cases}\dfrac{1}{4\sqrt{\pi}(t-s)}\cdot\displaystyle\sum_{k=-\infty}^{\infty}\frac{1-x+2k}{\sqrt{(t-s)}}\exp\left\{\frac{-(1-x+2k)^2}{4(t-s)}\right\} & \textit{for } t>s\\ 0 \textit{ for } t\le s\end{cases}$$

The kernel for $\nabla_x u$ will be

$$\frac{\partial K}{\partial x}(x,t;0,s)=$$

$$\begin{cases}\dfrac{1}{2\sqrt{\pi}(t-s)^{\frac{3}{2}}}\displaystyle\sum_{k=-\infty}^{\infty}\left(\frac{1}{2}-\left(\frac{(x+2k)}{2\sqrt{(t-s)}}\right)^2\right)\exp\left\{\frac{-(x+2k)^2}{4(t-s)}\right\} & \textit{if } t>s\\ 0 \quad \textit{if } t\le s\end{cases}$$

and at $y = 1$

$$\frac{\partial K}{\partial x}(x,t;1,s)=$$

$$\begin{cases}\dfrac{1}{2\sqrt{\pi}(t-s)^{\frac{3}{2}}}\displaystyle\sum_{k=-\infty}^{\infty}\left(-\frac{1}{2}+\left(\frac{(1-x+2k)}{2\sqrt{(t-s)}}\right)^2\right)\exp\left\{\frac{-(1-x+2k)^2}{4(t-s)}\right\} & \textit{for } t>s\\ 0 \quad \textit{for } t\le s\end{cases}$$

For $\nabla_t u$ at $y = 0$,

$$\frac{\partial K}{\partial t}(x,t;0,s)=$$

$$=\frac{1}{4\sqrt{\pi}}\sum_{k=-\infty}^{\infty}\left[\left(-\frac{3}{2}\right)\frac{(x+2k)}{(t-s)^{\frac{5}{2}}}+\frac{(x+2k)^3}{4(t-s)^{\frac{7}{2}}}\right]\cdot$$

$$\cdot\exp\left\{\frac{-(x+2k)^2}{4(t-s)}\right\}\textit{for } t>s, 0 \textit{ for } t\le s \textit{ or}$$

$$\frac{\partial K}{\partial t}(x,t;0,s)=\frac{1}{2\sqrt{\pi}(t-s)^2}$$

$$=\sum_{k=-\infty}^{\infty}\left[\frac{(x+2k)^3}{(4(t-s))^{\frac{3}{2}}}-\frac{3}{2}\frac{(x+2k)}{\sqrt{4(t-s)}}\right]\cdot$$

$$\cdot\exp\left\{\frac{-(x+2k)^2}{4(t-s)}\right\}\textit{for } t>s;\ 0 \textit{ for } t\le s.$$

The formulas for $y = 1$are similar to those at $y = 0$, so only the ones for $y = 0$ will be calculated from now on.

To see that $\int\frac{\partial K}{\partial t}=0$ and $\int\frac{\partial K}{\partial x}=0$ note that $\frac{\partial K}{\partial t}=-\frac{\partial K}{\partial s}$ and

$$\frac{\partial\theta_1(x-y,t-s)}{\partial x}=-\frac{\partial\theta_1(x-y,t-s)}{\partial y},$$

$$\frac{\partial\theta_1(x+y,t-s)}{\partial x}=\frac{\partial\theta_1(x+y,t-s)}{\partial y}\text{ so}$$

$$\frac{\partial K}{\partial x}=-\frac{\partial^2\theta_1(x-y,t-s)}{\partial y^2}-\frac{\partial^2\theta_1(x+y,t-s)}{\partial y^2}$$

$$=\frac{\partial\theta_1(x-y,t-s)}{\partial s}+\frac{\partial\theta_1(x+y,t-s)}{\partial s}$$

since $\theta_1(x \pm y, t-s)$ solves $\left(\frac{\partial\theta_1}{\partial s} + \frac{\partial^2\theta_1}{\partial y^2} = 0\right)$. By the definition and decay of G_1 and G_2 it is easy to see that $\int \frac{\partial K}{\partial t}\ (x,t;0,s)\ ds = 0$ and that $\int \frac{\partial K}{\partial x}\ (x,t;0,s)\ ds = 0$.

To show that $\frac{\partial K}{\partial x}$ and $\frac{\partial K}{\partial t}$ satisfy the decay estimates hypothesized in Theorem 2 one uses the uniform convergence of the series

$$\sum_{k=-\infty}^{\infty}\left(\frac{1}{2} - \frac{(x+2k)^2}{\left(2(t-s)^{\frac{1}{2}}\right)^2}\right)\exp\left\{\frac{-|x+2k|^2}{4(t-s)}\right\} \text{ and}$$

$$\sum_{k=-\infty}^{\infty}\left[\frac{(x+2k)^3}{\left(2(t-s)^{\frac{1}{2}}\right)^3} - \frac{3}{2}\frac{(x+2k)}{\left(2(t-s)^{\frac{1}{2}}\right)}\right]\cdot \exp\left\{\frac{-(x+2k^2)}{4(t-s)}\right\}.$$

In fact the series

$$\sum_{k=-\infty}^{\infty} P\left(\frac{x+2k}{2(t-s)^{\frac{1}{2}}}\right)\exp\left\{-\left(\frac{x+2k}{2(t-s)^{\frac{1}{2}}}\right)^2\right\}$$

will converge for any polynomial $P(z)$ of fixed degree m if $|t-s|^{\frac{1}{2}} \lesssim x$. An elementary calculation using Simpson's rule and the fact that $\int_{-\infty}^{\infty}\left(\frac{1}{2} - z^2\right)\ e^{-z^2}dz = 0$ gives decay $\leq \frac{C}{(t-s)^{3/2}}$ when $t-s$ is large, for the sum

$$\sum_{k=-\infty}^{\infty}\left(\frac{1}{2} - \frac{(x+2k)^2}{\left(2(t-s)^{\frac{1}{2}}\right)^2}\right)\exp\left\{\frac{-|x+2k|^2}{4(t-s)}\right\}$$

We can write the sum as

$$\frac{2\sqrt{t-s}}{6}\cdot 6\cdot\frac{1}{2\sqrt{t-s}}\sum_{j=-\infty}^{\infty}\sum_{\ell=(j-1)n}^{jn} P\left(\frac{x+2\ell}{2(t-s)^{\frac{1}{2}}}\right)\exp\left\{-\left(\frac{x+2\ell}{2(t-s)^{\frac{1}{2}}}\right)^2\right\}$$

$$= \frac{\sqrt{t-s}}{6}\sum_{j=-\infty}^{\infty}\frac{2}{\sqrt{t-s}}\left(P\left(\frac{x+2(j-1)n}{2\sqrt{t-s}}\right)\right)\cdot \exp\left\{-\left(\frac{x+2\ell}{2\sqrt{t-s}}\right)^2\right\}$$

$$+ 4P\left(\frac{x+2(j-1)n+2}{2\sqrt{t-s}}\right)\exp\left\{-\left(\frac{x+2(j-1)n+2}{2\sqrt{t-s}}\right)^2\right\}$$

$$+ 2P\left(\frac{x+2(j-1)n+4}{2\sqrt{t-s}}\right)\exp\left\{-\left(\frac{x+2(j-1)n+4}{2\sqrt{t-s}}\right)^2\right\}$$

$$+\ldots+4P\left(\frac{x+2jn-2}{2\sqrt{t-s}}\right)\exp\left\{-\left(\frac{x+2jn-2}{2\sqrt{t-s}}\right)^2\right\}$$

$$+P\left(\frac{x+2jn}{2\sqrt{t-s}}\right)\exp\left\{-\left(\frac{(x+2jn)}{2\sqrt{t-s}}\right)^2\right\}$$

$$= \frac{\sqrt{t-s}}{6}\frac{2}{\sqrt{t-s}}\sum_{j=-\infty}^{\infty} S_j(x,t-s) = \frac{\sqrt{t-s}}{6}\cdot$$

$$\left(\sum_{j=-\infty}^{\infty}\frac{2}{\sqrt{t-2}}S_j(x,t-2) - \int_{\frac{x+2(j-1)n}{2\sqrt{t-s}}}^{\frac{x+2jn}{2\sqrt{t-s}}} P(z)\exp\left\{\left(-z^2\right)\right\}dz\right)$$

because $\int_{-\infty}^{\infty} P(z)\exp\left\{-z^2\right\}dz = 0$ for both polynomials we are considering here. Now Simpson's rule gives the upper bound of

$$\max\left|Q(z)\exp\left\{-z^2\right\}\right|\cdot\frac{\left(\frac{n}{\sqrt{t-s}}\right)^5}{180(2n)^4} \lesssim \frac{M_j}{720\left(\sqrt{t-s}\right)^4}$$

where

$$\frac{x+2(j-1)n}{2\sqrt{t-s}} \leq z \leq \frac{x+2jn}{2\sqrt{t-s}} \text{ and } \sum_{j=-\infty}^{\infty} M_j < \infty.$$

$Q(z)$ is a polynomial of degree = $\deg P(z)\ +4$, $n = \sqrt{t-s}$.

Multiplying by $\frac{\sqrt{t-s}}{6}$ gives that

$$\left|\sum_{k=-\infty}^{\infty} P\left(\frac{x+2k}{2\sqrt{t-s}}\right)\exp\left\{-\left(\frac{x+2k}{2\sqrt{t-s}}\right)^2\right\} \leq \frac{M}{(t-s)^{\frac{3}{2}}}\right|$$

for $\sqrt{t-s}$ large, say $\sqrt{t-s}\ \gtrsim |x|$.

Similarly

$$\sum_{k=-\infty}^{\infty}\left[\frac{(x+2k)^3}{\left(2(t-s)^{\frac{1}{2}}\right)^3} - \frac{3}{2}\left(\frac{x+2k}{2(t-s)^{\frac{1}{2}}}\right)\right]\cdot \exp\left\{\frac{-(x+2k)^2}{4(t-s)}\right\}$$

will decay at a rate bounded by $\frac{C}{(t-s)^{3/2}}$, for $|t-s|^{\frac{1}{2}}, \gtrsim x$, say.

So writing

$$\frac{\partial K}{\partial x}(x,t;0,s) = \frac{1}{8\sqrt{\pi}(t-s)^{\frac{3}{2}}}\cdot$$

$$\sum_{k=-\infty}^{\infty}\left(2-\frac{(x+2k)^2}{(t-s)}\right)\exp\left\{\frac{-(x+2k)^2}{4(t-s)}\right\}$$

$$=\frac{1}{2\sqrt{\pi}}\frac{1}{x^3}\left(\frac{1}{\left(\frac{(t-s)}{x^2}\right)^{\frac{3}{2}}}\right)\sum_{k=-\infty}^{\infty}\left(\frac{1}{2}-\left(\frac{(x+2k)}{2\sqrt{t-s}}\right)^2\right).$$

$$\cdot\exp\left\{\frac{-(x+2k)^2}{4(t-s)}\right\}$$

and using the fact that $(x,t)\in T(I)$ implies that $x\sim\ell(I)$, one has the estimate

$$\left|\frac{\partial K}{\partial x}(x,t;0,s)\right|\le C\frac{1}{\ell(I)^3}\left(1+\frac{d_p((x,t);(0,s))}{\ell(I)}\right)^{-3}$$

and

$$\left|\frac{\partial K}{\partial t}(x,t;0,s)\right|\le C\frac{1}{\ell(I)^4}\left(1+\frac{d_p((x,t);(0,s))}{\ell(I)}\right)^{-4}$$

for all values of $t-s\ge 0$.

The same kind of analysis gives that

$$\left|\frac{\partial^2 K}{\partial x\partial s}(x,t;0,s)\right|\le C\frac{1}{\ell(I)^5}\left(1+\frac{d_p((x,t);(0,s))}{\ell(I)}\right)^{-5}$$

and

$$\left|\frac{\partial^2 K}{\partial t\partial s}(x,t;0,s)\right|\le C\frac{1}{\ell(I)^6}\left(1+\frac{d_p((x,t);(0,s))}{\ell(I)}\right)^{-6}.$$

This is the right smoothness for applying Theorem 2 with $H = M = 3$ for $\frac{\partial K}{\partial x}$ and $H = M = 4$ for $\frac{\partial K}{\partial t}$.

For the box domain in $\mathbb{R}^{d+1}$, $d>1$, similar results hold.

As in Doob [1] we take

$$R=\prod_{i=1}^{d}(0,1)\times(-\infty,t).$$

Here the kernel (see Hattemer [2] p. 129)

$$K(x,t;y,s)=\prod_{j=1}^{d}\left(G_1(x_j,t;y_j,s)+G_2(x_j,t;y_j,s)\right)$$

where

$$G_2(x_j,t;y_j,s)=\pm\frac{\partial G_1}{\partial y_j}(x_j,t;\delta,s)$$

(remember the sign corresponds to the inward normal) $\delta = 0$ or 1, if $y_j = 0$ or 1 and G_2 $(x_j,t;y_j,s)=0$ if $0<y_j<1$. So for $(y,s)\in\partial R$, say $(y,s)\in S_{j_0}$, then

$$K(x,t;y,s)=G_2(x_{j_0},t;\delta,s)\cdot\prod_{j\ne j_0}G_1(x_j,t;y_j,s)$$

and if $i\ne j_0$

$$\frac{\partial K}{\partial x_i}(x,t;y,s)=G_2(x_{j_0},t;\delta,s)\frac{\partial G_1(x_i,t;y_i,s)}{\partial x_i}\cdot$$

$$\prod_{j\ne i,j_0}G_1(x_j,t;y_j,s).$$

If $i=j_0$

$$\frac{\partial K}{\partial x_i}(x,t;y,s)=\frac{\partial G_2(x_{j_0},t;\delta,s)}{\partial x_{j_0}}\cdot\prod_{j\ne j_0}G_1(x_j,t;y_j,s).$$

To see that $\frac{\partial K}{\partial x_i}$ and $\frac{\partial K}{\partial t}$ have the mean value zero property note that the constant function

$$1=\int_{\partial R}K(x,t;y,s)\,dyds$$

so differentiating gives

$$0=\int_{\partial R}\frac{\partial K}{\partial x_i}(x,t;y,s)\,dyds$$

for $i=1,2,\ldots,d$. The decay and smoothness of the kernel allow differentiation under the integral sign.

To see that $\frac{\partial K}{\partial x_i}$ and $\frac{\partial K}{\partial t}$ have the right decay for Theorem 2, the estimates above can be used along with the following estimates for G_1:

$$G_1(x,t;y,s)=\frac{1}{2}\left[\theta_1(x-y,t-s)-\theta_1(x+y,t-s)\right]$$

$$=\frac{1}{2}\sum_{k=-\infty}^{\infty}\frac{1}{\sqrt{4\pi(t-s)}}\cdot$$

$$\left[\exp\left\{-\frac{|x-y+2k|^2}{4(t-s)}\right\}-\exp\left\{-\frac{|x+y+2k|^2}{4(t-s)}\right\}\right].$$

For $t-s$ large this sum can be interpreted as a Riemann-type sum taking

$$a=j=\frac{x+2k_0}{2(t-s)^{\frac{1}{2}}}\text{ and}$$

$$b=j+1=\frac{x+2k_0+2\ell}{2(t-s)^{\frac{1}{2}}}$$

(replacing $x\pm y$ by x here) and applying Simpson's Rule to each $\int_j^{j+1}$:

$$\int_a^b\exp\{-u^2\}du=\frac{b-a}{3n}\left[\exp\{-a^2\}\right.$$

$$+4\exp\left\{-\frac{(x+2k_0+2)^2}{4(t-s)}\right\}+2\exp\left\{-\frac{(x+2k_0+4)^2}{4(t-s)}\right\}$$

$$+\ldots+4\exp\left\{-\frac{(x+2k_0+2\ell-2)^2}{4(t-s)}\right\}+\exp\left\{-b^2\right\}\Bigg]+E_s^j$$

where $n=\ell\sim\sqrt{(t-s)}$ and E_s^j is the error in using Simpson's rule. Using

$$\int_{-\infty}^{\infty}\exp(-u^2)du-\int_{-\infty}^{\infty}\exp(-x^2)dx=0$$

and the fact that if each interval is shifted by $\frac{2}{4\sqrt{t-s}}$ and the two series are summed $j=-\infty$ to $+\infty$ and added together one has

$$6\sum_{k=-\infty}^{\infty}\frac{1}{2\sqrt{\pi}(t-s)^{\frac{1}{2}}}\exp\left\{-\frac{|x+2k|^2}{4(t-s)}\right\}+2\sum_{j=-\infty}^{\infty}E_s^j$$

This gives

$$\begin{aligned}&|G_1(x,t;y,s)|\\&=\frac{1}{6}\Bigg|\sum_{k=-\infty}^{\infty}\frac{6}{\sqrt{4\pi(t-s)}}\exp\left\{-\frac{|x-y+2k|^2}{4(t-s)}\right\}\\&\quad-\int_{-\infty}^{\infty}\frac{2}{\sqrt{\pi}}\exp\left\{-u^2\right\}du\\&\quad-\sum_{k=-\infty}^{\infty}\frac{6}{\sqrt{4\pi(t-s)}}\exp\left\{-\frac{|x+y+2k|^2}{4(t-s)}\right\}\\&\quad+\int_{-\infty}^{\infty}\frac{2}{\sqrt{\pi}}\exp\left\{-x^2\right\}dx\Bigg|\\&\lesssim\sum_{j=-\infty}^{\infty}\left|E_s^j\right|\lesssim\sum\frac{M_j\cdot(1)^5}{180(t-s)^{\frac{4}{2}}}\le\frac{1}{|t-s|^2}\sum M_j\,.\end{aligned}$$

The nature of the function $\exp\{-x^2\}$ implies that $\sum_{j=-\infty}^{\infty}M_j=M<\infty$.

Altogether $|G_1(x_j,t;y_j,s)|\lesssim\frac{1}{|t-s|^2}$ for $|t-s|$ large. For $|t-s|$ small the somewhat more conservative estimate of $\ell(I)^{-1}$ or $|x_j-y_j|^{-1}$ must be used. As above each derivative in a space variable $\frac{\partial}{\partial x_j}$ or $\frac{\partial}{\partial y_j}$ will increase the exponent of decay by $\frac{1}{2}$ and each time derivative will send it up by 1 giving the estimates:

$$\left|\frac{\partial K}{\partial x_i}(x,t;y,s)\right|\lesssim\frac{1}{\ell(I)^3}\left(1+\frac{d_p(x_{j_0},t;0,s)}{\ell(I)}\right)^{-3}\cdot$$

$$\prod_{j\neq j_0}\frac{1}{\ell(I)}\left(1+\frac{d_p(x_j,t;0,s)}{\ell(I)}\right)^{-1}$$

$$=\frac{1}{\ell(I)^{d+2}}\cdot\left(1+\frac{d_p(x,t;0,s)}{\ell(I)}\right)^{-(d+2)}\quad\text{and}$$

$$\left|\frac{\partial K}{\partial t}(x,t;y,s)\right|\lesssim\frac{1}{\ell(I)^{d+3}}\left(1+\frac{d_p(x,t;0,s)}{\ell(I)}\right)^{-(d+3)}$$

So putting $M=d+2$ and $H=d+2$ for $\frac{\partial K}{\partial x_i}$ and $M=H=d+3$ for $\frac{\partial K}{\partial t}$ means that the estimates needed to apply Theorem 2 are valid.

References

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