EXAM II  EE-270 Random Variables & Stochastic Processes  November 10, 2010

Exam Due: Thursday November 18, 8:30 am

TAKE HOME EXAM
You are expected to work independently
Do all problems

NAME ________________________________

Please write clearly and underline your answers wherever you can.

1. ________

2. ________

3. ________

4. ________

5. ________

6. ________
1. (10 points) 

**Discrete random variable**

Consider the experiment of tossing a coin. The experiment is specified by the sample space $S = \{h, t\}$. Define a random variable $X \to \mathbb{R}$. For the two values of the random variable, say 0 and 1, let it have probabilities $p$ and $q$. Do the following:

(i) Plot $F_X(x)$.

(ii) Plot $f_X(x)$ and give an expression for it.

(iii) Find the *probability function* $P\{X=0\}$ and $P\{X=1\}$. 
2. (10 points)

**Bernoulli and Binomial random variable**

As we know, a discrete RV with two possible outcomes is called a Bernoulli RV. Its distribution $F_X(x)$ is called the Bernoulli distribution,

$$P\{X = 1\} = p, P\{X = 0\} = q$$

$Y$ is said to be a binomial RV with parameters $n$ and $p$ if $Y$ takes on values $0, 1, 2, \ldots, n$ with

$$P\{Y = k\} = \binom{n}{k} p^k q^{n-k}$$

(Note that $P\{Y = k\}$ is often called the *probability function* $P_Y(y)$ of the discrete RV $Y$.)

(i) Plot $F_Y(y)$, $f_y(y)$ and the probability function $P_Y(y)$.

Also write the expression for $f_y(y)$. 

3. (10 points)
Function of a RV
\[ Y = g(X), F_Y(y), f_Y(y) \]

For the ideal diode, we have
\[ Y = \begin{cases} 
X, & \text{if } x \geq 0 \\
0, & \text{if } x < 0 
\end{cases} \]

Given that \( f_X(x) \) is N(0,1), determine and plot \( F_Y(y) \) and \( f_X(x) \).
4. (20 points)

**Binomial and Bernoulli random variables**

X is said to be a Bernoulli RV if it takes the values 1 or 0, with $p$ representing the probability of success. If there are $n$ independent Bernoulli trials, and $Y$ represents the favorable outcomes (k successes in n trials) then, $Y$ is said to be a Binomial RV.

Consider the problem of transmitting information in blocks of 6 ($n=6$). This then corresponds to words of 6 bits. Let $p$ denote the probability of error in a bit and let $p = 0.1$. Assume that we define error in a block only when there are 2 or more bit errors. Determine the probability of block error.
Bayes Theorem

We consider the problem of car warranties. Cars sold by a company are assembled in one of four possible locations. The four locations supply 20%, 24%, 25% and 31% respectively. Since a buyer does not know where the car was assembled, one assumes the probabilities of the purchased car being from each of the 4 plants as 0.20, 0.24, 0.25 and 0.31.

Data collected by the company shows the following:
\[
\begin{align*}
P(\text{claim—Plant 1}) &= 0.05 \\
P(\text{claim—Plant 2}) &= 0.11 \\
P(\text{claim—Plant 3}) &= 0.03 \\
P(\text{claim—Plant 4}) &= 0.08
\end{align*}
\]

Determine the following:

(i) \( P(\text{claim}) \)

(ii) If now a claim is made on the warranty of the car, how does the prior probabilities in paragraph 1 above change? That is, what are the a posteriori probabilities for each of the 4 plants?

(iii) Determine the a posteriori probabilities if no claim is made on the warranty.

(iv) Fill in the table:

\[
\begin{array}{|c|c|c|}
\hline
 & \text{Prior Probabilities} & \text{Posterior Probabilities} \\
\hline
\text{Plant I} & 0.200 & \text{Claim} \\
\text{Plant II} & 0.240 & \text{No Claim} \\
\text{Plant III} & 0.250 & \\
\text{Plant IV} & 0.310 & \\
\text{Sum} & & \\
\hline
\end{array}
\]

(v) Note that Plant II had the largest claim rate and Plant III had the smallest claim rate. Having calculated the a posteriori probabilities, comment on whether that is expected.
6. (25 points)  
**Bayes Theorem**

In this problem, we essentially go over the material and examples 4-18, 4-19 in the text on pages 102-105.

(a) **Total Probability, continuous version**

For the case where we have a Bernoulli random variable \( H \) that represents the event \{head in one toss\} of a coin, and where the probability of of heads is not a number, but a random variable \( P \), we have as shown in Eq. (4-80), that

\[
P(H) = \int_{-\infty}^{+\infty} P(H|P = p)f_P(p)\,dp
\]

\( P(H) \) is calculated as in Eq. (4-82).

(i) What type of random variables are \( H \) and \( P \)? That is, continuous or discrete?

(ii) Explain why \( P\{H|P = p\} = p \).

(iii) We get for \( P(H) \)

\[
P(H) = \int_{0}^{1} p f(p)\,dp
\]

In the language of random variables, what statistic has been determined here?

(iv) Now consider the case where \( H \) represents the event of \{k heads in n tosses\}. Give the expression for \( P(H) \)?

(b) **A Posteriori Distribution**

Assume now that we have gathered more data, that is, tossed the coin \( n \) times and \( H \) now represents the event \{k heads in n tosses\}. The a posteriori distribution (pdf) can be calculated and is shown in Eq.(4-86) (and is known as the beta pdf).

(i) Now if we were to toss the coin once, what is the \( P(\text{getting a head}) \)? As in item (iv) above, what statistic are we calculating here?

(ii) The answer to \( P(\text{getting a head}) \) is given in Eq. (4-88) and is,

\[
P(\text{getting a head}) = \frac{k + 1}{n + 1}
\]

What we have here is the Bayesian estimate of the probability of “getting a head” given that the prior distribution was uniform. How does this relate to the frequency description of probability?

(c) **Another example**

Construct another example. That is, a priori probability of a RV associated with an expriment, data gathering, a posterior probability of that RV.