The Calderón reproducing formula converges unconditionally in $L^p$

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Abstract.

We prove that a very general form of the Calderón reproducing formula converges in $L^p(w)$, for all $1 < p < \infty$, whenever $w$ belongs to the Muckenhoupt class $A_p$.

1. Introduction.

In this note we demonstrate a result so well known that it (apparently) has not been proved before.

The Calderón reproducing formula is a familiar object. It is frequently defined this way. Suppose that $\psi$ and $\phi$, both belonging to $C_0^\infty(\mathbb{R}^d)$, are real, radial, have supports contained in $\{x : |x| \leq 1\}$, have integrals equal to 0, and satisfy

$$\int_0^\infty \hat{\psi}(y\xi) \hat{\phi}(y\xi) \frac{dy}{y} = 1 \quad (1.1)$$

for all $\xi \neq 0$. Then, if $f$ lies in a “suitable” test class of functions defined on $\mathbb{R}^d$, one can write

$$f(x) = \int_{\mathbb{R}^{d+1}_+} (f \ast \psi_y(t)) \phi_y(x - t) \frac{dt \; dy}{y},$$

(1.2)

where equation (1.2) is typically said to hold “in the sense of distributions.” (We have used some standard notations: $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times (0, \infty)$, $h_y(x) = y^{-d}h(x/y)$, and “$\ast$” is convolution.) A justification of (1.2) typically runs like this. Take the Fourier transform of the right-hand side of (1.2). Formally, it is

$$\hat{f}(\xi) \int_0^\infty \hat{\psi}(y\xi) \hat{\phi}(y\xi) \frac{dy}{y},$$

which, by (1.1), equals $\hat{f}(\xi)$ for all non-zero $\xi$. QED!

Actually, things are not quite that bad. The Plancherel and Fourier convolution theorems imply that

$$\int_{\mathbb{R}^{d+1}_+} |f \ast \psi_y(t)|^2 \frac{dt \; dy}{y}$$

(1.3)

and

$$\int_{\mathbb{R}^{d+1}_+} |f \ast \phi_y(t)|^2 \frac{dt \; dy}{y}$$

(1.4)

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are both bounded by constants times \( \|f\|_2^2 \). Knowing this, it is not hard to establish
\[
\int_{\mathbb{R}^d} |f(x)|^2 \, dx = \int_{\mathbb{R}^{d+1}_+} (f \ast \psi_y(t)) (\bar{f} \ast \phi_y(t)) \frac{dt \, dy}{y}
\]
for all \( f \in L^2 \). Polarization then lets us assert
\[
\int_{\mathbb{R}^d} f(x) h(x) \, dx = \int_{\mathbb{R}^{d+1}_+} (f \ast \psi_y(t)) (h \ast \phi_y(t)) \frac{dt \, dy}{y} \tag{1.5}
\]
for all \( f \) and \( h \) in \( L^2 \), where our bounds on (1.3) and (1.4) imply that the right-hand side of (1.5) is absolutely convergent. (We note that, although the convolution with \( h \) in (1.5) looks strange, it’s okay: \( \phi \) is even, so \( \phi_y(x-t) = \phi_y(t-x) \).)

That would seem to give an answer to how (1.2) converges. Since \( L^2 \) is dense in any reasonable space, one can use (1.2) to approximate any \( f \)—considered as a linear functional—to arbitrary precision. The trouble with this is that, in many applications (e.g., involving integral operators or atomic decompositions, and in weighted spaces), one often wants to use (1.2) to get information about \( f \) considered as a function. This usually entails cutting the integral in (1.2) into infinitely many pieces, estimating the pieces somehow, and summing the pieces to get estimates on \( f \). I suppose it would be possible to do all of this purely in the context of distributions, but I can’t recall ever seeing anybody do this. People always treat (1.2) as if it represented \( f \) as a function, but rarely ask about the sense in which it does this.

We said “rarely,” which is not the same as “never.” A natural way to approach the convergence of (1.2) is in terms of approximating subsets of \( \mathbb{R}^{d+1}_+ \). If \( A \subset \mathbb{R}^{d+1}_+ \) is measurable and has compact closure \( \overline{A} \) contained in \( \mathbb{R}^{d+1}_+ \), we can define
\[
f_{(A)}(x) \equiv \int_A (f \ast \psi_y(t)) \phi_y(x-t) \frac{dt \, dy}{y}
\]
for all \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \). This is a bounded function, and it makes sense to ask whether the sequence \( f_{(A_j)} \) converges to \( f \) in such-and-such manner (in \( L^p \), say) as the sets \( A_j \rightharpoonup \mathbb{R}^{d+1}_+ \).

The answer to this question can be ‘no.’ The integral of any \( f_{(A)} \) is 0; therefore, we can’t expect convergence in the \( L^1 \) norm. If \( f \) is a constant function, every \( f_{(A)} \) is identically 0; therefore, we don’t even get weak-star convergence in \( L^\infty \).

The problem of convergence has been considered for sequences of “rectangular” sets \( A_j \),
\[
A_j = \{(x,y) : |x| \leq R_j, \ 0 < \epsilon_j < y < T_j\},
\]
where \( \epsilon_j \searrow 0 \) and \( R_j, T_j \nearrow \infty \) as \( j \to \infty \). In [Daub] it is shown that, if \( f \in L^2 \), then \( f_{(A_j)} \to f \) in \( L^2 \); and that, if \( f \in L^2 \cap L^\infty \), then \( f_{(A_j)}(x) \to f(x) \) at every point where \( f \) is continuous. In [FJW], the authors show convergence in \( L^2 \) for sets of the form
\[
\tilde{A}_j = \{(x,y) : 0 < \epsilon_j < y < T_j\},
\]
under the assumption that \( f \in L^2 \).
In this paper we prove that $f(A_j) \to f$ in $L^p$, for very general increasing sequences of sets \{A_j\}, for all $1 < p < \infty$, and for pairs of functions $\phi$ and $\psi$ that satisfy very weak decay and smoothness estimates. But more is true. The convergence even holds in $L^p(w)$, where $w$ belongs to the Muckenhoupt $A_p$ class. We must emphasize that this convergence holds for all $f \in L^p(w)$, and not just for a dense subclass.

We need to define some terms. The first one is familiar. A weight $w$ is said to belong to $A_p (1 < p < \infty)$ if it is non-trivial and if

$$\sup_Q \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^{1/p} \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'} \, dx \right)^{1/p'} < \infty,$$

where the supremum is over all cubes $Q \subset \mathbb{R}^d$, $p' = p/(p-1)$ is $p$’s dual index, and $|Q|$ denotes $Q$’s Lebesgue measure.

The next two definitions are not so familiar, but they should not give the reader any difficulties.

We will say that \{E_j\} is a compact-measurable exhaustion of $\mathbb{R}^{d+1}$ if each $E_j$ is a measurable subset of $\mathbb{R}^{d+1}$, each $E_j$ has compact closure $\overline{E_j}$ contained in $\mathbb{R}^{d+1}$, $E_j \subseteq E_{j+1}$ for all $j$, and $\bigcup_j E_j = \mathbb{R}^{d+1}$. Note that we do not require $\overline{E_j}$ to be contained in $E_{j+1}$, but we do insist that $E_j$ be bounded and that it stay away from $\partial \mathbb{R}^{d+1}$.

If $0 < \beta \leq 1$ and $\epsilon > 0$, we define $C(\beta, \epsilon)$ to be the family of functions $\phi : \mathbb{R}^d \to \mathbb{R}$ such that:

a) for all $x \in \mathbb{R}^d$,

$$|\phi(x)| \leq (1 + |x|)^{-d-\epsilon};$$

b) for all $x$ and $x'$ in $\mathbb{R}^d$,

$$|\phi(x) - \phi(x')| \leq |x - x'|^{\beta} ((1 + |x|)^{-d-\epsilon} + (1 + |x'|)^{-d-\epsilon});$$

and c) $\int \phi \, dx = 0$.

In section 2 we prove:

**Theorem 1.** Let $\phi$ and $\psi$ be radial functions, both positive multiples of functions in $C(\beta, \epsilon)$, and normalized so that

$$\int_0^\infty \hat{\psi}(y \xi) \hat{\phi}(y \xi) \frac{dy}{y} = 1 \quad (1.6)$$

for all $\xi \neq 0$. Let \{E_j\} be any compact-measurable exhaustion of $\mathbb{R}^{d+1}$. If $w \in A_p (1 < p < \infty)$ and $f \in L^p(w)$ then, for all $j$,

$$f_{(E_j)}(x) \equiv \int_{E_j} (f \ast \psi_y(t)) \phi_y(x - t) \frac{dt \, dy}{y}$$

defines a function in $L^p(w)$, and the sequence \{f_{(E_j)}\} converges to $f$ in $L^p(w)$.

The proof consists of a few simple estimates, one Littlewood-Paley inequality, and a little functional analysis.

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We will use $C$ to denote a constant that might change from occurrence to occurrence, but not so as to make the resulting inequalities trivial. Whenever we say “measurable,” we mean “Lebesgue measurable.” The word “weight” will always mean a non-negative function in $L_{\text{loc}}^1(\mathbb{R}^d)$. Whenever we refer to $A_p$, we assume that $1 < p < \infty$. We use $L^p(w)$ to denote the weighted space $L^p(\mathbb{R}^d, w \, dx)$ and we use $L^p(dx)$ to refer to ordinary, unweighted $L^p$—i.e., $L^p(\mathbb{R}^d, 1 \, dx)$.

We wish to express our deep gratitude to David Cruz-Uribe and José Martell, for making us aware of Theorem 4, and to David Ullrich, for needling us on the cavalier attitude people often take toward the convergence of Calderón’s formula (“Standard manipulations . . . left to the reader . . . trivial”).

2. Lemmas and theorems.

We recall three familiar facts about $A_p$ when $1 < p < \infty$.

1. If $w \in A_p$ then $\| M(f) \|_{L^p(w)} \leq C \| f \|_{L^p(w)}$ for all $f \in L^p(w)$, where $M(\cdot)$ is the Hardy-Littlewood maximal operator.

2. If $w \in A_p$ then $w^{1-p} \in A_{p'}$.

3. If $w \in A_p$, there is a $q_0 > 1$ such that $w \in A_{p/q}$ for all $1 < q < q_0$.

Lemma 1. If $1 < p < \infty$ and $w \in A_p$ then $(1 + |x|)^{-d} \in L^p(w)$.

Proof. If $w \in A_p$ and $Q$ is any cube, then $M(\chi_Q) \in L^p(w)$. But $M(\chi_Q)$ is comparable to $(1 + |x|)^{-d}$.

Lemma 2. If $w \in A_p$ and $f \in L^p(w)$ then $|f(x)|(1 + |x|)^{-d} \in L^1(dx)$.

Proof.

\[
\int_{\mathbb{R}^d} |f(x)|(1 + |x|)^{-d} \, dx = \int_{\mathbb{R}^d} |f(x)| \, w^{1/p} (1 + |x|)^{-d} \, w^{-1/p} \, dx
\]

\[
\leq \left( \int_{\mathbb{R}^d} |f(x)|^p \, w \, dx \right)^{1/p} \left( \int_{\mathbb{R}^d} (1 + |x|)^{-dp'} \, w^{1-p'} \, dx \right)^{1/p'} < \infty.
\]

Corollary 1. Suppose that $f \in L^p(w)$ for some $w \in A_p$. If $\phi$ and $\psi$ belong to $C_{(\beta, \epsilon)}$ and $A$ is a measurable set with compact closure contained in $\mathbb{R}^{d+1}_+$, then

\[
f_{(A)}(x) \equiv \int_A (f * \psi_y(t)) \phi_y(x - t) \frac{dt \, dy}{y}
\]

makes sense for all $x \in \mathbb{R}^d$ and defines a function in $L^p(w)$.

Proof. Lemma 2 implies that $f * \psi_y(t)$ is defined and bounded for $(t, y) \in A$. Therefore $|f_{(A)}(x)| \leq C(1 + |x|)^{-d-\epsilon}$ for all $x$.

Lemma 3. If $\phi \in C_{(\beta, \epsilon)}$ then, for all $\xi$,

\[
\int_0^\infty |\hat{\phi}(y\xi)|^2 \frac{dy}{y} < \infty.
\]
Proof. By looking at things in polar coordinates, it is easy to see that the integral from 1 to infinity converges. The only possible problem occurs near 0. Thus, it will be enough to show that \(|\hat{\phi}(\xi)| \leq C|\xi|^\tau\) for small \(|\xi|\), for some \(\tau > 0\). For \(0 < \alpha \leq 1\), define \(C_\alpha\) to be the family of functions \(\eta\) supported in \(\{x: |x| \leq 1\}\), having integral equal to 0, and satisfying

\[|\eta(x) - \eta(x')| \leq |x - x'|^\alpha\]

for all \(x\) and \(x'\). In [W] it is proved that every \(\phi \in C_{(\beta, \epsilon)}\) can be written as a rapidly-converging sum of dilates of functions belong to some \(C_\alpha\), in the following precise sense: For all \(0 < \beta \leq 1\) and \(\epsilon > 0\), there exist \(0 < \alpha \leq 1\) and \(\delta > 0\) and a constant \(C\) such that, for all \(\phi \in C_{(\beta, \epsilon)}\), we can find a sequence of functions \(\{\eta^{(j)}\}_{j=0}^\infty \subset C_\alpha\) for which

\[\phi(x) = C \sum_{j=0}^\infty 2^{-j\delta}(\eta^{(j)})(2jx)\]

But every \(\eta \in C_\alpha\) satisfies

\[|\hat{\eta}(\xi)| \leq C \min(|\xi|, 1)\]

Let \(0 \leq |\xi| \leq 1\) and write

\[\hat{\phi}(\xi) = C \sum_{j: 2^j|\xi| \leq \sqrt{1}} 2^{-j\delta}\hat{\eta}^{(j)}(2^j \xi) + C \sum_{j: 2^j|\xi| > \sqrt{1}} 2^{-j\delta}\hat{\eta}^{(j)}(2^j \xi)\]

\[= (I) + (II)\].

But \(|(I)| \leq C_\delta |\xi|^{1/2}\), while \(|(II)| \leq C|\xi|^{\delta/2}\).

By noting that the left-hand side of (2.2) is invariant under the change of variable \(y \mapsto ty\) \((t > 0)\), we obtain:

**Corollary 2.** Let \(\phi\) be as in the preceding lemma. There is a \(C = C(\beta, \epsilon, d)\) such that, for all \(0 \leq A < B < \infty\) and all \(\xi \neq 0\),

\[\int_A^B |\hat{\phi}(y\xi)|^2 \frac{dy}{y} \leq C.\]

**Remark.** It is not hard to see that \(\eta \in C_\alpha\) implies

\[|\hat{\eta}(\xi)| \leq C \min(|\xi|, |\xi|^{-b})\]

for some positive \(b\), from which one can obtain the estimate

\[|\hat{\phi}(\xi)| \leq C \min(|\xi|^{\delta_1}, |\xi|^{-\delta_2})\]

for some \(\delta_1 > 0\).
Definition 1. If $\phi \in C(\beta, \epsilon)$ and $|f(x)|(1 + |x|)^{-d-\epsilon} \in L^1(dx)$, we set

$$g_\phi(f)(x) \equiv \left( \int_{0}^{\infty} |f * \phi_y(x)|^2 \frac{dy}{y} \right)^{1/2}.$$ 

If $A \subset \mathbb{R}^{d+1}_+$ is measurable, we define

$$g_{\phi,A}(f)(x) \equiv \left( \int_{y \in (0,\infty): (x,y) \in A} |f * \phi_y(x)|^2 \frac{dy}{y} \right)^{1/2},$$

for every $x$ such that $\{y \in (0, \infty) : (x,y) \in A\}$ is measurable.

Theorem 1 will follow from the next result, whose (short) proof we give at the end of the paper.

Theorem 2. Let $w \in A_p$, and suppose that $0 < \beta \leq 1$ and $\epsilon > 0$. There is a constant $C = C(w, p, \beta, \epsilon)$ such that, for all $\phi \in C(\beta, \epsilon)$ and $f \in L^p(w)$,

$$\int_{\mathbb{R}^d} ((g_\phi(f))(x))^p w \, dx \leq C \int_{\mathbb{R}^d} |f(x)|^p w \, dx.$$

Here is how Theorem 2 will give us Theorem 1. Suppose that $\phi$ and $\psi$ satisfy the hypotheses of Theorem 1. (Lemma 3 ensures that there is no problem about the convergence of the integral in (1.6).) If $A$ is a measurable set with compact closure contained in $\mathbb{R}^{d+1}_+$, and $f \in L^p(w)$, define $f(A)$ by (2.1). Let $h \in L^{p'}(w)$ satisfy $\|h\|_{L^{p'}(w)} \leq 1$, and put $H \equiv hw$. Notice that $H \in L^{p'}(w^{1-p'})$, which implies that we can convolve $H$ with functions in $C(\beta, \epsilon)$. We can write:

$$\left| \int_{\mathbb{R}^d} f(A)(x) h(x) \, w \, dx \right| = \left| \int_{A} (f * \psi_y(t)) (H * \phi_y(t)) \frac{dt \, dy}{y} \right|$$

$$\leq \int_{\mathbb{R}^d} g_{\psi,A}(f)(t) g_{\phi}(H)(t) \, dt$$

$$\leq \|g_{\psi,A}(f)\|_{L^p(w)} \|g_{\phi}(H)\|_{L^{p'}(w^{1-p'})},$$

using the factoring trick from Lemma 2. Theorem 2 will give us

$$\|g_{\phi}(H)\|_{L^{p'}(w^{1-p'})} \leq C \|H\|_{L^{p'}(w^{1-p'})} = C \|h\|_{L^{p'}(w)} \leq C,$$

which lets us infer

$$\|f(A)\|_{L^p(w)} \leq C \|g_{\psi,A}(f)\|_{L^p(w)}.$$

Let $\{E_j\}$ be any compact-measurable exhaustion of $\mathbb{R}^{d+1}_+$. If $j < k$ then

$$f(E_k) - f(E_j) = f(E_k \setminus E_j),$$
and therefore

\[ \|f(E_k) - f(E_j)\|_{L^p(w)} \leq C\|g_{\psi,E_k \setminus E_j}(f)\|_{L^p(w)}. \]

But \( f \in L^p(w) \) implies \( g_{\psi}(f) \in L^p(w) \). Since \( g_{\psi}(f) < \infty \) w.a.e., Dominated Convergence (on \((0, \infty)\)) implies that \( g_{\psi,E_k \setminus E_j}(f) \to 0 \) w.a.e. as \( k \) and \( j \) go to infinity. It is trivial that \( g_{\psi,E_k \setminus E_j}(f) \leq g_{\psi}(f) \). Another application of Dominated Convergence implies that \( g_{\psi,E_k \setminus E_j}(f) \to 0 \) in \( L^p(w) \). Therefore, the sequence \( \{f(E_j)\} \) is Cauchy in \( L^p(w) \).

Here we could say “polarization” again, but we believe the following argument, while a touch longer, might be more illuminating.

Let \( h \in L^2(dx) \). We have that

\[ \int_{\mathbb{R}^{d+1}} |f \ast \psi_y(t)| |h \ast \phi_y(t)| \frac{dt dy}{y} \leq \int_{\mathbb{R}^d} g_{\psi}(f)(t) g_{\phi}(h)(t) dt < \infty, \]

implying that

\[ \lim_j \int_{\mathbb{R}^d} f(E_j)(x) h(x) dx = \lim_j \int_{E_j} (f \ast \psi_y(t)) (h \ast \phi_y(t)) \frac{dt dy}{y} \]

\[ = \lim_j \int_{\tilde{A}_j} (f \ast \psi_y(t)) (h \ast \phi_y(t)) \frac{dt dy}{y}, \]

where \( \tilde{A}_j = \{(x, y) : 2^{-j} < y < 2^j\} \). But, for each \( j \), the function

\[ G_j(x) \equiv \int_{\tilde{A}_j} (f \ast \psi_y(t)) \phi_y(x - t) \frac{dt dy}{y} \]

belongs to \( L^2(dx) \cap L^1(dx) \), with Fourier transform

\[ \hat{G}_j(\xi) = \hat{f}(\xi) \int_{2^{-j}}^{2^j} \hat{\psi}(y \xi) \hat{\phi}(y \xi) \frac{dy}{y}. \]

For each \( j \),

\[ \int_{\mathbb{R}^d} G_j(x) h(x) dx = \int_{\tilde{A}_j} (f \ast \psi_y(t)) (h \ast \phi_y(t)) \frac{dt dy}{y}. \]

(We can say this because all the integrals involved are absolutely convergent.) Corollary 2 implies that \(|\hat{G}_j(\xi)| \leq C|\hat{f}(\xi)|\), independent of \( j \) and \( \xi \), while our normalization (1.6)
implies that $\hat{G}_j(\xi) \to \hat{f}(\xi)$ almost-everywhere. Therefore $G_j \to f$ in $L^2(dx)$, implying that

$$
\int_{\mathbb{R}^d} f(x) h(x) \, dx = \lim_j \int_{\mathbb{R}^d} G_j(x) h(x) \, dx \\
= \lim_j \int_{\mathcal{A}_j} (f \ast \psi_y(t)) (h \ast \phi_y(t)) \, \frac{dt \, dy}{y} \\
= \lim_j \int_{E_j} (f \ast \psi_y(t)) (h \ast \phi_y(t)) \, \frac{dt \, dy}{y} \\
= \lim_j \int_{\mathbb{R}^d} f(E_j)(x) h(x) \, dx \\
= \int_{\mathbb{R}^d} T(f)(x) h(x) \, dx,
$$

where the last equation follows because $T(f)$ is also the $L^2(dx)$ limit of the $f(E_j)$'s.

One way to prove Theorem 2 would be to first phrase it in terms of vector-valued singular integral operators. The proof given below is more direct (it completely avoids good-$\lambda$ inequalities and the theory of vector-valued operators) and, we believe, it yields a better estimate on the size of $g_\phi(f)$. It’s also the proof we thought of first.

If we replace $g_\phi(\cdot)$ with the classical Littlewood-Paley $g$-function, then Theorem 2 has been known for a long time (see [SegWh] and [GunWh]). In [StrTor], a version of Theorem 2 is proved for Littlewood-Paley functions based on the Gaussian heat kernel. A domination argument is then used to extend the result to $g$-functions defined via convolution kernels in the Schwartz class (see also [Kur] for an earlier result involving “real-variable” square functions; again, the kernels are assumed to be Schwartz). We believe Theorem 2 is the first formal statement of this result for kernels satisfying essentially minimal smoothness and decay estimates.

Actually, something apparently much stronger than Theorem 2 is true. Suppose that $f$ is such that $|f(x)|(1 + |x|)^{-d-\epsilon} \in L^1(dx)$. For $(t, y) \in \mathbb{R}_+^{d+1}$, set

$$
\tilde{A}_{(\beta, \epsilon)}(f)(t, y) \equiv \sup_{\psi \in C_{(\beta, \epsilon)}} |f \ast \psi_y(t)|.
$$

For $x \in \mathbb{R}^d$, define

$$
\tilde{g}_{(\beta, \epsilon)}(f)(x) \equiv \left( \int_0^\infty (\tilde{A}_{(\beta, \epsilon)}(f)(x, y))^2 \frac{dy}{y} \right)^{1/2}.
$$

We note the trivial fact that $g_\phi(f) \leq \tilde{g}_{(\beta, \epsilon)}(f)$ pointwise.

In [W] the following result is proved:

**Theorem 3.** For all $1 < p \leq 2$, all $0 < \beta \leq 1$, and $\epsilon > 0$, there is a constant $C = C(\beta, \epsilon, p, d)$ such that, for all weights $v$, and for all $f$ having $|f(x)|(1 + |x|)^{-d-\epsilon} \in L^1(dx)$,

$$
\int_{\mathbb{R}^d} (\tilde{g}_{(\beta, \epsilon)}(f)(x))^p v \, dx \leq C \int_{\mathbb{R}^d} |f(x)|^p M(v) \, dx,
$$

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where $M(\cdot)$ denotes the Hardy-Littlewood maximal operator.

The expression “better estimate,” used above, refers to the fact that the analogue of Theorem 3 fails for singular integral operators.

Theorem 3 will immediately yield Theorem 2, via the following extrapolation result:

**Theorem 4.** Let $f$ and $g$ be two functions such that, for some $p_0 > 1$ and all $1 < q < p_0$, there is a $C = C(q)$ so that, for all weights $v$,

$$\int_{\mathbb{R}^d} |f(x)|^q v \, dx \leq C(q) \int_{\mathbb{R}^d} |g(x)|^q M(v) \, dx. \quad (2.3)$$

Then, for all $1 < p < \infty$ and $w \in A_p$, there is a $C = C(w, p)$ such that

$$\int_{\mathbb{R}^d} |f(x)|^p w \, dx \leq C(w, p) \int_{\mathbb{R}^d} |g(x)|^p w \, dx.$$

Theorem 4 is implicit in many arguments (e.g., [Duo], pp. 141–142), and is well-known to experts in extrapolation theory. Unlike Theorem 3, Theorem 4 has a very quick proof, which we give here for the sake of completeness.

Let $w \in A_p$, and pick $1 < q < \min(p, p_0)$ such that $w \in A_{p/q}$. Put $r = p/q > 1$. Let $h \in L^{r'}(w)$ satisfy $\|h\|_{L^{r'}(w)} = 1$ and be such that

$$\int_{\mathbb{R}^d} |f(x)|^q |h(x)| w \, dx = \|f\|^q_{L^p(w)}.$$

Using (2.3), we can write:

$$\int_{\mathbb{R}^d} |f(x)|^q |h(x)| w \, dx \leq C \int_{\mathbb{R}^d} |g(x)|^q M(hw) \, dx$$

$$= C \int_{\mathbb{R}^d} |g(x)|^q w^{q/p} M(hw) w^{-q/p} \, dx$$

$$\leq C \left( \int_{\mathbb{R}^d} |g(x)|^p w \, dx \right)^{1/r} \left( \int_{\mathbb{R}^d} (M(hw))^{r'} w^{-r'q/p} \, dx \right)^{1/r'}.$$

Now,

$$\frac{-r'q}{p} = \frac{-r'}{r} = 1 - r',$$

implying that $w^{-r'q/p}$ will belong to $A_{r'}$ if $w \in A_r$—which is true, by our choice of $q$. Therefore

$$\int_{\mathbb{R}^d} (M(hw))^{r'} w^{-r'q/p} \, dx \leq C \int_{\mathbb{R}^d} |h(x)|^{r'} w^{r'} w^{1-r'} \, dx = 1,$$

proving the result.

We can use the preceding analysis to extend the definition of $f(A)$ to arbitrary measurable $A \subset \mathbb{R}^{d+1}_+$. It is easy to see that, if $f \in L^p(w)$ for $w \in A_p$, and $\{E_j\}$ is any compact-measurable exhaustion, then $\{f_{(A \cap E_j)}\}$ has an $L^p(w)$ limit that is independent of the sequence $\{E_j\}$. We can call the limit $f(A)$. This agrees with our earlier definition of $f(A)$ when $\overline{A}$ is a compact subset of $\mathbb{R}^{d+1}_+$. The following corollary is now easy to prove.
Corollary 3. Let \( \phi, \psi, \) and \( w \) be as in the hypotheses of Theorem 1. Suppose that \( R^d+1 = \bigcup_j A_j \), where the sets \( A_j \) are measurable and disjoint. If \( f \in L^p(w) \) then

\[
f = \sum_j f_{(A_j)},
\]

where the infinite series converges unconditionally in \( L^p(w) \).

References.


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